# SINGLE VALUED AND SET VALUED INTEGRALS IN LOCALLY CONVEX SPACES

BY

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TABLE OF CONTENTS

ACKNOWLEDGMENTS.....iii

ABSTRACT.....v

CHAPTERS

I INTRODUCTION.....1

II SINGLE VALUED INTEGRALS.....9

II.1 Integration of Vector Valued Functions.....9
II.2 Integration With Respect to a Vector Valued Measure.....27

III SET VALUED INTEGRALS.....44

III.1 The Spece of Subsets of X.....44
III.2 Set Valued Set Functions.....58
III.3 Integration.....66
III.4 Comparison With Other Integrals.....77
III.5 The Weak Integral.....82
III.6 Englarging the Class of Integrable Functions.....84
III.7 Integration With Respect to a Vector-Set Valued Measure.....89

IV CONCLUSION.....93

BIBLIOGRAPHY.....94

BIOGRAPHICAL SKETCH.....98

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In this work, four integrals are studied, two single valued and the other two set valued. The single valued integrals are generalizations of the Pettis and Bartle-Dunford-Schwartz-Lewis integrals to the case that the underlying measure is only finitely additive. Also in the latter case, the integrals take their values in a locally convex space. The central role played by strong additivity for the absolute continuity of these integrals is exhibited. Vitali type convergence theorems are established for the integrals of each kind. For the Pettis generalization, sufficient conditions for the convergence of Pettis-cauchy sequences are given. The Bartle-Dunford-Schwartz-Lewis section concludes with several theorems on representation of operators on the spaces $B(S,\Sigma)$, $C(S)$, and $L_1$.

Next, two set valued integrals are studied. The first is a new set valued integral for functions taking their values in the space of

bounded convex subsets of a complete Hausdorff locally convex space. This integral is a direct generalization of the Bochner integral and is seen to coincide with the Aumann and Debrew integrals under proper conditions. Preliminary to the introduction of the integral is a study of infinite series of sets in the space of bounded convex subsets, and a study of set valued set functions. Here also, the concept of strong additivity is visited. The integral itself follows a standard development parallel to that of the Bochner integral. The comparison to the Aumann and Debrew integrals follows and then the introduction of a weak integral. Finally, a different sort of weakening of the defining hypothesis is explored to enlarge the class of integrable functions. The second set valued integral was introduced by J.K. Brooks in 1968 and here we give a kind of monotone convergence theorem for that integral.

# CHAPTER I
# INTRODUCTION

In this paper we will generalize integration theory in three directions. First, measures will be assumed only to be finitely additive with specific results for the countably additive case indicated. Second, with the exception of section II.1 on functionally defined integrals of vector valued functions, the integrals will take values in Hausdorff locally convex spaces or in the space of subsets of a Hausdorff locally convex space. The last direction is the case that the integral is set valued.

The reader familiar with modern integration theory will recognize generalizations of single valued integrals presented in part I to be extensions of the work of Pettis [Pe38], Bartle, Dunford and Schwartz [Ba55] and Lewis [Le70]. These integrals are defined functionally as the solutions of the equations

$$x^*(x) = \int x^*(f)d\mu \text{ and } x^*(x) = \int f d(x^*\mu),\ x^* \varepsilon \Xi^*,$$

depending on whether the function is vector valued and the measure scalar (Pettis) or vice-versa (Bartle-Dunford-Schwartz and Lewis). Though the integrals have no countable additivity to inherit (since the measures used will be assumed only to be finitely additive), it is seen here that a consequence of a functional definition of an integral

is that the integral is strongly additive. This is due to the Orlicz-Pettis theorem and the fact that indefinite integrals of functions in $L_1$ are measures of bounded variation. It is upon this fact that the absolute continuity of these integrals rests. The role of strong additivity is increasingly apparent in the literature of vector measure theory (as shown in Diestel and Uhl [Di77]) and will be revisited in the Chapter III.

The convergence theory of the generalization of the Pettis integral to the finitely additive case serves as a model for the convergence theory of the generalization of the Bartle-Dunford-Schwartz-Lewis integral. The convergence theorems presented in section 4 of Pettis [Pe38] still hold in the finitely additive setting and we give here a more direct proof of the density of simple functions in the space of totally measurable Pettis integrable functions over a space of finite variation. To round off the convergence theory of the Pettis integral, we present a Vitali Convergence Theorem appropriate in this setting. The remainder of the Pettis section is devoted to an analysis of how the Pettis integral differs from the strong integral and what additional assumptions are needed to guarantee the convergence of Pettis-Cauchy sequences of Pettis integrable functions.

In 1972, Lewis [Le70] presented a generalization of the Bartle-Dunford-Schwartz integral which extended the theory to handle functions with values in a Hausdorff locally convex topological vector space. The celebrated representation theorem of weakly compact operators on C(S) as vector measures was extended to this more general

setting and we include the result and the consequential Dunford-Pettis property for completeness. Also a representation theorem for operators on $L_1$ is given in terms of a vector measure.

Set valued functions, or correspondences, have been studied for over 70 years beginning perhaps with Felix Hausdorff [Hau62] in 1922. The first studies of correspondences centered around continuity and include those by R.L. Moore [Mo25] in 1925 and L.S. Hill [Hi27] in 1927. Continuity, semicontinuity, compactness, connectedness, and contraction properties continued to be studied by many including Kuratowski [Kur32] in 1932, Choquet [Ch47] in 1947, and Bourbaki [Bo60] in 1960. The fixed point behavior of correspondences has been studied by many authors including Kakutani [Ka41] in 1941 and Michael [Mi56a, Mi56b] in 1956.

Set valued integration theory began properly in 1942 with the definition of the first set valued integral by C.E. Rickart [Ric42]. The roots of Rickart's integral are found in an integral defined in 1930 by Kolmogorov [Ko30] in which the first glimmer of set valued integration theory is seen. There Kolmogorov defined the integral for functions whose values were sets of real numbers even though the integral itself is a single point. Garret Birkhoff [Bi35] defined an integral in 1935 which also allowed integrands to be set valued while the integral is a single point. Birkhoff's integral also generalizes the Bochner integral. Before Rickart presented his integral, important ideas about topologies on spaces of sets had been studied by E.H. Moore and H.L. Smith [Moo22] in 1922, by Felix Hausdorff [Hau62] in 1927, and by John von Neumann [vo35] in 1935. Rickart defined a

type of convergence of sets analogous to the Hausdorff metric convergence of sets. The topology of this convergence has been demonstrated by Drewnowski [Dr76] to be generated by a uniformity on the space of subsets. The basic idea of closeness is that of considering two sets to be close if each is contained in the sum of the other and a particular neighborhood of zero, the neighborhood being the measure of closeness. All such pairs of sets for a given neighborhood define an element of the uniformity. I have used a mixture of ideas of Hausdorff, Rickart, and Drewnowski to define the same topology on the space of nonempty bounded subsets of a locally convex space by defining a family of Hausdorff like semi-distances between sets, one for each seminorm of the LCS. I have shown that the space of subsets is complete if the LCS is complete and has a countably generated neighborhood basis. Using his analogue of Hausdorff convergence of sets, Rickart also examined ideas of convergence of series of sets so that countable additivity could be formulated for set valued measures; in particular, he showed his integral was countably additive in this sense. He called this mode of summability unconditional summability. Birkhoff also presented an idea of convergence of series of sets that is defined in terms of selections of the sets. That is, for a sequence of sets he considered sequences of single vectors, one from each set. The sequence of sets is summable if each selection of the sequence forms an unconditionally convergent series, and in that case, the sum of the sets is the set of the sums of all the selections. After M. Sousa [So85], I refer to this mode of summability of sets as unconditional selection

summability. I have shown that Birkhoff's unconditional selection summability of sets and Rickart's unconditional summability are the same for sequences of bounded sets in a complete LCS. Sousa obtained this result in Banach spaces. Both forms of summability have natural applications in the theory. The former is more general, while the latter has the advantage of using ordinary series of vectors. I have given a Pettis type theorem for the equivalence of weak and strong countably additivity for set valued set functions by using Birkhoff's unconditional selection summability and a generalization by A.P. Roberts [Ro69] of the Orlicz-Pettis theorem for locally convex spaces. An explosion of interest in set valued integration occurred after the definition in 1965 by R.J. Aumann [Au65] of a new set valued integral. Similar to Birkhoff's selections of series of sets, Aumann's integral definition involves selections of the set valued functions to be integrated. A single valued function is a selection of a set valued function if its value at every point is contained in the value of the set valued function at the same point. Any function having a Bochner integrable selection is Aumann integrable and the Aumann integral is the set of integrals of all the integrable selections. Though this is a natural generalization of the Bochner integral, it gives rise to an indefinite integral that is not necessarily an absolutely continuous set function. This is because structure of a set valued function and of selections of that function can be very different. Even very simple set valued functions can have an enormous number of selections. Consequently, the set of Aumann integrable functions is very large. Mathematical economists, who had

already been studying correspondences and set valued measures or multimeasures, quickly became interested in Aumann's integral as a way to unite two differing but related points of view: one of sets of commodities associated to individual economic agents and one of sets of commodities associated to sets or coalitions of those agents. If the individuals form a set and the coalitions of agents form a sigma algebra of subsets of the set of agents, then the two points of view are those of set valued functions and set valued measures sharing a measurable space. A complete study of this relationship should involve an integral for the set valued functions and a Radon-Nikodym type derivative for the set valued measures. The Radon-Nikodym challenge was taken by Debrew and Schmeidler [De71] in 1971 for the Aumann integral when they produced a Radon-Nikodym derivative for set valued measures having their set values within the positive orthant of a Euclidean space. In 1972, Artstein [Ar72] extended the existence of the derivative to the general Euclidean valued case. Owing to the classical structure of my integral, I hope to take up the challenge of finding more general Radon-Nikodym derivatives for set valued measures, perhaps in Banach spaces. In fact since my integral agrees with the Aumann integral for reflexive Banach spaces, it already has the Radon-Nikodym theorem for the Euclidean case. Debrew [De65] also defined an integral of his own in his 1965 paper and showed that it was equivalent to Aumann's integral. Debrew's integral was an integral for functions whose values were compact-convex subsets of a real Banach space. Using Radstrom's [Ra52] embedding theorem, the space of nonempty compact convex sets is considered as a subset of

another real Banach space and the integration is carried out in the bigger space using the Bochner integral. In 1940 G.B. Price [Pr40] noted that the theory of integration in a Banach should be easily carried to the metric space of compact convex subsets with the Hausdorff metric. The integral I defined parallels this aspect of the Debrew integral by actually being a direct generalization of the Bochner integral, and as a consequence, my integral contains the Debrew integral as a special case. The set valued integrals mentioned so far are by no means a complete list. Another set valued integral defined by J.K. Brooks [Br68a, Br68b] in 1968 treated the case of a set valued measure rather than a set valued integrand. I have provided a kind of monotone convergence theorem for this integral. Set valued integrals have also found their way into statistics in papers by Dvoretzky [Dv50], Kudo [Ku54] and Richter [Ri63], into control theory in papers by Hermes [He68] and Jacobs [Ja69], and into game theory in a paper by Schmiedler [Sc71]. While enjoying the spotlight of integration theory in economic studies, general set valued measure theory was also being studied as a field of its own. Best evidence of this is an elaborate study by Drewnowski [Dr76] in 1972 where among other things he established Vitali-Hahn-Saks and Nikodym boundedness type theorems and also existence of nonnegative control measures. Drewnowski uses a generalization by Hormander [Ho54] of Radstrom's embedding theorem for many of the results. The concept of strong additivity of a set valued measure plays an important role in Drewnowski's study. I have explored the relationship of strong additivity to unconditional summability and

having defined weak set valued integrals, I have used strong additivity for their convergence theorems.

CHAPTER II
SINGLE VALUED INTEGRALS

## II.1. Integration of Vector Valued Functions

S denotes a set with an algebra $\Sigma$ of subsets on which an additive scalar valued function $\mu$ is defined. $\Xi$ denotes a Banach space and $\Xi^*$ its dual. The symbol $v(\mu,E)$ denotes the total variation of $\mu$ on a set $E\varepsilon\Sigma$. $v^*(\mu,A)$ denotes the outer $v(\mu)$-measure $\inf_{A \subset E \varepsilon \Sigma} v(\mu,E)$ of any subset A of S. A set $E \subset S$ is a null set if $v^*(\mu,E) = 0$ $\mu$ is strongly additive if $\mu(E_n) \to 0$ whenever $(E_n)$ is disjoint in $\Sigma$. Absolute continuity of a measure $\lambda$ with respect to $\mu$ means $\lim_{v(\mu,E)\to 0} \lambda(E) = 0$ and is also denoted by $\lambda << \mu$. A function $f:S \to \Xi$ is a null function if $v^*(\mu,\{s\varepsilon S: |f(s)|>\varepsilon\})=0$ for all $\varepsilon>0$. A function $f:S \to \Xi$ is a simple function if it is constant on each of a finite number of sets in $\Sigma$ or if it differs from such a function by a null function. A sequence $(f_n)$ of functions converges to a function f in $\mu$-measure if $\lim_{n\to\infty} v^*(\mu,\{s \varepsilon S: |f_n(s) - f(s)|>\varepsilon\}) = 0$ for every $\varepsilon>0$, and is denoted by $f_n - \mu \to f$. A function f on S is totally $\mu$-measureable if there is a sequence $(f_n)$ of simple functions on S converging in measure to f. A function f on S is measurable if its restruction to each set $E \varepsilon \Sigma$ of finite variation is totally $\mu$-measurable. A function f on S is weakly (totally) $\mu$-measurable if $x^*f$ is (totally) measurable for every $x^* \varepsilon \Xi^*$. A simple function $f = \sum_n x_n 1_{E_n}$ is integrable if $x_n = 0$ whenever $v(\mu,E_n) = \infty$. The

integral of f over a set E in $\Sigma$ is given by

$$\int_E f d\mu = \sum_n \mu(EE_n)x_n.$$

A function $f:S \to \Xi$ is Bochner integrable if there is a sequence $(f_n)$ of simple integrable functions converging in measure to f with the property

$$\lim_{n,m\to\infty} \int_S |f_n - f_m| dv(\mu) = 0.$$

The integral of f over a set E of $\Sigma$ is given by

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

The integral is independent of the particular sequence $(f_n)$ used. A useful property of an integrable function f is the existence of a sequence $(f_n)$ of simple integrable functions such that $\lim_n \int_S |f_n - f| dv(\mu) = 0$. The space of Bochner integrable functions is denoted by $L_1(S,\Sigma,\mu,\Xi)$ and is a normed space with norm $|f|_1 = \int_S |f| dv(\mu)$. A function $f:S \to \Xi$ is Dunford integrable if $x^* f \in L_1$ for every $x^* \in \Xi^*$. The functional $x^* \to \int_E x^* f d\mu$ is the Dunford integral of f over $E \in \Sigma$. If f is Dunford integrable, we define

$$N(f) = \sup_{|x^*|\leq 1} \int_S |x^* f| dv(\mu)$$

which is a nonnegative subadditive possibly infinite functional with $N(\alpha f) = |\alpha| N(f)$ for $\alpha$ scalar. Note also that $|x^* f|_1 \leq |x^*| N(f)$. If f is Dunford integrable and the Dunford integral over $E \in \Sigma$ is an element of x for every $E \in \Sigma$, then it is called the Pettis integral of f.

$D(S,\Sigma,\mu,\Xi)$ denotes the linear space of Dunford integrable functions and $P(S,\Sigma,\mu,\Xi)$ denotes the subspace of Pettis integrable

functions. $D_1$ and $P_1$ denote the subspaces of D and P respectively for which $N(f)<\infty$.

Theorem 1. For $f\in P$, the set function $E \to \int_E f d\mu$ on $\Sigma$ is strongly additive and $\mu$-continuous.

Proof: Let $E, F \in \Sigma$ such that $EF = \emptyset$. Since the Bochner integral is additive on $\Sigma$, we have for each $x^* \in \Xi$

$$x^*\int_{E\cup F} f d\mu = \int_{E\cup F} x^* f d\mu = \int_E x^* f d\mu + \int_F x^* f d\mu = x^*(\int_E f d\mu + \int_F f d\mu).$$

That $\int_{E\cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$ now follows from the fact that $\Xi^*$ separates points of $\Xi$. Let $\phi = \int f d\mu$. If $(E_n)$ is a disjoint sequence in $\Sigma$, then so is $(E_{n_k})$ for any subsequence $(n_k)$. Let $x^* \in \Xi^*$ be arbitrary. Then $x^*\phi$ is a measure of bounded variation and in particular it is strongly additive. Thus $\Sigma_n x^*\phi(E_{n_k})$ is convergent. By the Orlicz-Pettis Theorem it follows that $\Sigma_n \phi(E_n)$ is unconditionally convergent hence $\phi$ is strongly additive. It then follows that $\{x^*\phi : |x^*| \le 1\}$ is uniformly strongly additive. Since $\lim_{\nu(\mu,E)\to 0} x^*\phi(E) = 0$ for every $x^*$ it follows from Theorem 1 of Brooks [Br73] that $\lim_{\nu(\mu,E)\to 0} x^*\phi(E) = 0$ uniformly for $|x^*| \le 1$. Since $|\phi(E)| = \sup_{|x^*|\le 1} |x^*\phi(E)|$, it follows that $\phi$ is $\mu$-continuous.

Corollary 2. For $f \in P$, the vector measure $E \to \int_E f d\mu$ is bounded and has relatively weakly compact range (see [Ri44, Theorem 2.4] and [Di77, Corollary I.5.3]).

Lemma 3. For $f \in P$, we have $\sup_{E\in\Sigma} |\int_E f d\mu| \le N(f) \le 4 \sup_{E\in\Sigma} |\int_E f d\mu|$. Thus $P_1 = P$ by Corollary 2.

Proof: Let $\phi(E) = \int_E f d\mu$. For $x^* \in \Xi^*$ we have

$$|x^*f|_1 = v(x^*\phi, S) \leq 4 \sup_{E \in \Sigma} \left| \int_E x^* f d\mu \right| \leq 4 \sup_{E \in \Sigma} \left| \int_E f d\mu \right|;$$

hence

$$N(f) = \sup_{|x^*| \leq 1} |x^*f|_1 \leq 4 \sup_{E \in \Sigma} \left| \int_E f d\mu \right|.$$

Also, $|\phi(E)| = \sup_{|x^*| \leq 1} |x^*\phi(E)| \leq N(f)$; hence $\sup_{E \in \Sigma} |\phi(E)| \leq N(f)$.

Theorem 4. If $f \in P$ is $\mu$-measurable and $v(\mu, S) < \infty$, then there is a sequence $(f_n)$ of simple integrable functions such that $N(f_n - f) \to 0$.

Proof: Let $(g_n)$ be a sequence of simple (integrable since $v(\mu, S) < \infty$) functions converging in measure to $f$. For each $k$ choose $n_k$ and $E_k \in \Sigma$ such that $v(\mu, E_k) < 1/k$ and $|g_{n_k}(s) - f(s)| < 1/k$ for $s \in S - E_k$. Let $f_k(s) = g_{n_k}(s)$ for $s \in S - E_k$ and $f_k(s) = 0$ for $s \in E_k$.
For $|x^*| \leq 1$ we have

$$\int_S |x^*(f - f_k)| dv(\mu) = \int_{S - E_k} |x^*(f - g_{n_k})| dv(\mu) + \int_{E_k} |x^* f| dv(\mu)$$

$$\leq v(\mu, S)/k + \int_{E_k} |x^* f| dv(\mu)$$

$$= v(\mu, S)/k + v\left( \int x^* f d\mu, E_k \right)$$

$$\leq v(\mu, S)/k + 4 \sup_{F \subset E_k, F \in \Sigma} \left| \int_F x^* f d\mu \right|$$

$$\leq v(\mu, S)/k + 4 \sup_{F \subset E_k, F \in \Sigma} \left| \int_F f d\mu \right|.$$

Hence

$$N(f - f_k) = \sup_{|x^*|\le 1}\int_S |x^*(f - f_k)|dv(\mu)$$

$$\le v(\mu,S)/k + 4\sup_{F \subset E_k, F \in \Sigma}\left|\int_F f d\mu\right|.$$

Since $\int f d\mu$ is $\mu$-continuous and $v(\mu,E_k) \to 0$, it follows that $\lim_{k\to\infty} N(f - f_k) = 0$.

The space of functions $f \in D_1$ such that $N(f - f_n) \to 0$ for some sequences $(f_n)$ of simple functions will be denoted by $P_s(S,\Sigma,\mu,\Xi)$.

Theorem 5. Let $f:S \to \Xi$. If there is a sequence $(f_n)$ in $D_1$ with $N(f_n - f_m) \to 0$ as $n,m \to \infty$ and $x^* f_n - \mu \to x^* f$ for each $x^*$, then $f \in D_1$ and $N(f_n - f) \to 0$.

Proof: For $x^* \in \Xi^*$ we have

$$\int_S |x^*(f_n - f_m)|dv(\mu) \le |x^*|N(f_n - f_m);$$

hence $(x^* f_n)$ is cauchy in $L_1$. Since $x^* f_n - \mu \to x^* f$, it follows that $x^* f \in L_1$ and $x^* f_n \to x^* f$ in $L_1$. Let $\varepsilon > 0$ and choose $n_\varepsilon$ such that $N(f_n - f_m) < \varepsilon$ for $n,m \ge n_\varepsilon$. Then for $n \ge n_\varepsilon$ and $|x^*| \le 1$ we have

$$\int_S |x^*(f_n - f)|dv(\mu) = \lim_{m\to\infty}\int_S |x^*(f_n - f_m)|dv(\mu) \le \varepsilon$$

thus $N(f_n - f) \le \varepsilon$.

Corollary 6. Let $(f_n)$, $f$ be as above. If $f_n \in P$ for each $n$, then $f \in P$ and $\int_E f d\mu = \lim_{n\to\infty}\int_E f_n d\mu$ for each $E \in \Sigma$.

Proof. From the proof of Theorem 5 we have $x^*f \in L_1$ and $x^*f_n \to x^*f$ in $L_1$ for each $x^*$, and $f \in D_1$. Let $E \in \Sigma$. Since $|\int_E(f_n - f_m)d\mu| \leq N(f_n - f_m) \to 0$ it follows that the limit $\lim_{n\to\infty}\int_E f_n d\mu$ exists. If this limit is denoted by $x_E$, then for each $x^*$ we have

$$x^*x_E = x^*(\lim_{n\to\infty}\int_E f_n d\mu) = \lim_{n\to\infty}\int_E x^*f_n d\mu = \int_E x^*f d\mu;$$

hence $f \in P_1$ and

$$\int_E f d\mu = x_E = \lim_{n\to\infty}\int_E f_n d\mu.$$

Corollary 7. $P_s \subset P$.

Corollary 8. For $f \in P_s$ the set functin $E \to \int_E f d\mu$ has relatively compact range.

Proof: Let $\lambda(E) = \int_E f d\mu$. If $f$ is simple, then by Corollary 2 the range of $\lambda$ is a bounded subset of a finite dimensional subspace of $\Xi$ hence is relatively compact. In general let $(f_n)$ be a sequence of simple integrable functions such that $N(f_n - f) \to 0$, and let $\lambda_n(E) = \int_E f_n d\mu$. By Lemma 3 we have

$$\sup_{E \in \Sigma}|\lambda_n(E) - \lambda(E)| \to 0.$$

Corollary 9. $f \in P_s$ iff there is a sequence $(f_n)$ of simple integrable functions such that $\lim_{n,m\to\infty}N(f_n - f_m) = 0$ and $x^*f_n - \mu \to x^*f$ for every $x^* \in \Xi^*$.

Proof: If $f \in P_s$, then there is a sequence $(f_n)$ of simple integrable functions such that $N(f_n - f) \to 0$. Clearly $N(f_n - f_m) \to 0$ as $n,m \to \infty$.

Also, for each $x^*$ we have

$$\int_S |x^*(f_n - f)| dv(\mu) \leq |x^*| N(f_n - f) \to 0$$

and it follows from the Vitali convergence theorem that $x^* f_n - \mu \to x^* f$. Conversely assume $N(f_n - f_m) \to 0$ and $x^* f_n - \mu \to x^* f$ for each $x^*$. By Theorem 5 and Corollary 6, $f \in P_1$ and $N(f_n - f) \to 0$ thus $f \in P_S$.

Theorem 10. Assume $\Sigma$ is a $\sigma$-algebra and $v(\mu,S)<\infty$ and let $f:S \to \Xi$. If $(f_n)$ is a sequence in P such that the limit $\lim_{n\to\infty} \int_E f_n d\mu$ exists for each $E \in \Sigma$ and such that $x^* f_n - \mu \to x^* f$ for each $x^*$, then $f \in P$ and

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

Proof: Let $x^* \in \Xi^*$. Since $\lim_{n\to\infty} \int_E x^* f_n d\mu$ exists for each $E \in \Sigma$, it follows from Theorem 3 of Brooks [Br70] that

$$\lim_{v(\mu,E) \to 0} \int_E x^* f_n d\mu = 0$$

uniformly for $n = 1,2,\ldots$. Since

$$\int_E |x^* f_n| dv(\mu) = v(\int x^* f_n d\mu, E) \leq 4 \sup_{F \subset E, F \in \Sigma} \left| \int_F x^* f_n d\mu \right|,$$

it follows that

$$\lim_{v(\mu,E) \to 0} \int_E |x^* f_n| dv(\mu) = 0$$

uniformly for $n = 1,2,\ldots$. Since $x^* f_n - \mu \to x^* f$ and $v(\mu,S)<\infty$, it follows from the Vitali Convergence Theorem that $x^* f \in L_1$ and

$x^* f_n \to x^* f$ in $L_1$. For $x^* \in \Xi^*$ we have

$$x^*(\lim_{n\to\infty}\int_E f_n d\mu) = \lim_{n\to\infty}\int_E x^* f_n d\mu = \int_E x^* f d\mu.$$

Hence $f \in P$ and $\int_E f d\mu = \lim_{n\to\infty}\int_E f_n d\mu$.

Theorem 11 (Vitali Convergence Theorem for P and $v(\mu,S)<\infty$).

Let $(f_n)$ be a sequence in P and $f:S \to \Xi$. Then $f \in P$ and $N(f_n - f) \to 0$ iff

(1) $x^* f_n - \mu \to x^* f$ uniformly for $|x^*| \le 1$, and

(2) $\lim_{v(\mu,E)\to 0}\int_E f_n d\mu = 0$ uniformly for $n = 1,2,\ldots$.

Proof: Assume the conditions (1) and (2) hold. We first show that $\lim_{n,m\to\infty} N(f_n - f_m) = 0$. Let $\varepsilon > 0$. Since

$$\int_E |x^* f_n| dv(\mu) = v(\int x^* f_n d\mu, E) \le 4\sup_{F\subset E, F\in\Sigma}\left|\int_F x^* f_n d\mu\right|$$

$$\le |x^*| 4\sup_{F\subset E, F\in\Sigma}\left|\int_F f_n d\mu\right|$$

it follows from (2) that we may choose a number $\delta > 0$ such that $v(\mu,E) < \delta$ implies $\int_E |x^* f_n| dv(\mu) < \varepsilon$ for all $|x^*| \le 1$ and $n = 1,2,\ldots$. From (1) we choose an integer $n_\varepsilon$ such that for $n,m \ge n_\varepsilon$ there is a set $E_{nm} \in \Sigma$ with $v(\mu,E_{nm}) < \delta$ and $|x^*(f_n(s) - f_m(s))| \le \varepsilon$ for $s \in S - E_{nm}$ and $|x^*| \le 1$. Thus for $|x^*| \le 1$,

$$|x^*(f_n - f_m)|_1 =$$

$$\int_{E_{nm}} |x^*(f_n - f_m)|dv(\mu) + \int_{S - E_{nm}} |x^*(f_n - f_m)|dv(\mu)$$

$$< 2\varepsilon + \varepsilon v(\mu,S).$$

Hence

$$N(f_n - f_m) = \sup_{|x^*| \le 1} |x^*(f_n - f_m)|_1 < \varepsilon(2 + v(\mu,S))$$

for $n,m \ge n_\varepsilon$ and we have $N(f_n - f_m) \to 0$. By Theorem 5 and Corollary 6 it follows that $f \in P$ and $N(f_n - f) \to 0$. Conversely, assume $f \in P$ and $N(f_n - f) \to 0$. Let $\varepsilon > 0$ and choose $n_\varepsilon$ such that $N(f_n - f) < \varepsilon$ for $n \ge n_\varepsilon$. For each $n \ge n_\varepsilon$ and $x^* \in \Xi^*$, the nonnegative function $|x^*(f_n - f)|$ is in $L_1$; hence there is a sequence $(h_{nkx^*})_{k=1}^{\infty}$ of nonnegative simple integrable functions converging in $\mu$-measure to $|x^*(f_n - f)|$ such that

$$\int_S h_{nkx^*} dv(\mu) \to \int_S |x^*(f_n - f)|dv(\mu)$$

as $k \to \infty$. Then for $n \ge n_\varepsilon$, $|x^*| \le 1$, and large $k$, we have $\int_S h_{nkx^*} dv(\mu) < 2\varepsilon$ and without loss of generality we assume this true for all $k$. For $\gamma > 0$, let $E_{nkx^*\gamma} = \{s \in S : h_{nkx^*}(s) > \gamma\}$. Since $h_{nkx^*}$ is a simple function we may assume $E_{nkx^*\gamma} \in \Sigma$. We have

$$\gamma v(\mu, E_{nkx^*\gamma}) = \int_{E_{nkx^*\gamma}} \gamma dv(\mu) \le \int_S h_{nkx^*} dv(\mu) < 2\varepsilon$$

for $n \ge n_\varepsilon$ and $|x^*| \le 1$; hence $v(\mu, E_{nkx^*\gamma}) < 2\varepsilon/\gamma$.

Since $h_{nkx}{}^* - \mu \to |x^*(f_n - f)|$ there is for each k large, a set $F_{nkx}{}^*{}_\gamma$ in $\Sigma$ such that $v(\mu, F_{nkx}{}^*{}_\gamma) < \varepsilon/\gamma$ and $|h_{nkx}{}^* - |x^*(f_n - f)|| < \gamma$ for $s \in S - F_{nkx}{}^*{}_\gamma$. Let $G_{nkx\gamma} = E_{nkx}{}^*{}_\gamma \bigcup F_{nkx}{}^*{}_\gamma$. For k large and $s \in S - G_{nkx}{}^*{}_\gamma$ we have

$$|x^*(f_n(s) - f(s))| \le h_{nkx}{}^*(s) + |h_{nkx}{}^*(s) - |x^*(f_n(s) - f(s)|| < 2\gamma.$$

Let $\rho, \delta > 0$ and choose $\gamma, \varepsilon$ such that $2\gamma < \rho$ and $3\varepsilon/\gamma < \delta$. Then for $n \ge n_\varepsilon$, $|x^*| \le 1$ and large k we have $|x^*(f_n(s) - f(s))| < \rho$ except for $s \in G_{nkx}{}^*{}_\gamma$ where $G_{nkx}{}^*{}_\gamma \in \Sigma$ and $v(\mu, G_{nkx}{}^*{}_\gamma) < \delta$. It follows that $x^* f_n - \mu \to x^* f$ uniformly for $|x^*| \le 1$. Thus (1) is proved. By a familiar argument we have $\lim_{v(\mu,E) \to 0} \int_E |x^* f| dv(\mu) = 0$ uniformly for $|x^*| \le 1$; hence there is a $\delta > 0$ such that $\int_E |x^* f| dv(\mu) < \varepsilon$ for $v(\mu,E) < \delta$ and $|x^*| \le 1$. Then for such E and $x^*$ we have

$$\int_E |x^* f_n| dv(\mu) \le \int_E |x^*(f_n - f)| dv(\mu) + \int_E |x^* f| dv(\mu) < 2\varepsilon$$

for $n \ge n_\varepsilon$. For each $n = 1, 2, \ldots n_\varepsilon - 1$, choose $\delta_n$ such that $\int_E |x^* f_n| dv(\mu) < \varepsilon$ for $v(\mu,E) < \delta_\varepsilon$ and $|x^*| \le 1$. The theorem is complete if $\delta$ is replaced by $\min_{1 \le n \le n_\varepsilon - 1} \{\delta_n, \delta\}$.

Lemma 11a. For each $f \in P_s$ and $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ such that $v(\mu, E_\varepsilon) < \infty$ and $\sup_{|x^*| \le 1} \int_{S - E_\varepsilon} |x^* f| dv(\mu) < \varepsilon$.

Proof: Choose a simple integrable function $f_\varepsilon$ such that $N(f - f_\varepsilon) < \varepsilon$. Let $E_\varepsilon$ be the union of those sets $f_\varepsilon^{-1}(s)$ for $f_\varepsilon(s) \ne 0$. Since $f_\varepsilon$ is integrable we have $v(\mu, E_\varepsilon) < \infty$. For each $x^*$ we have

$$\int_{S - E_\varepsilon} |x^* f| dv(\mu) \leq \int_{S - E_\varepsilon} |x^*(f - f_\varepsilon)| dv(\mu) + \int_{S - E_\varepsilon} |x^* f_\varepsilon| dv(\mu)$$

$$= \int_{S - E_\varepsilon} |x^*(f - f_\varepsilon)| dv(\mu).$$

Hence for $|x^*| \leq 1$ we have

$$\sup_{|x^*| \leq 1} \int_{S - E_\varepsilon} |x^* f| dv(\mu) \leq N(f - f_\varepsilon) < \varepsilon.$$

Theorem 12 (Vitali Convergence Theorem for $P_s$). Let $(f_n)$ be a sequence in $P_s$ and $f: S \to \Xi$. Then $f \in P_s$ and $N(f_n - f) \to 0$ iff

(1) $x^* f_n - \mu \to x^* f$ uniformly for $|x^*| \leq 1$,

(2) $\lim_{v(\mu, E) \to 0} \int_E f_n d\mu = 0$ uniformly for $n = 1, 2, \ldots$, and

(3) for each $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ with $v(\mu, E_\varepsilon) < \infty$ and

$$\int_{S - E_\varepsilon} |x^* f_n| dv(\mu) < \varepsilon \text{ for each } |x^*| \leq 1 \text{ and } n = 1, 2, \ldots.$$

Proof: Assume conditions (1), (2), and (3) hold. Again the first aim is to show $N(f_n - f_m) \to 0$. Let $\varepsilon > 0$ and let $E_\varepsilon \in \Sigma$ be given by (3). For $|x^*| \leq 1$ we have

$$|x^*(f_n - f_m)|_1 = \int_{S - E_\varepsilon} |x^*(f_n - f_m)| dv(\mu) + \int_{E_\varepsilon} |x^*(f_n - f_m)| dv(\mu)$$

$$\leq 2\varepsilon + \int_{E_\varepsilon} |x^*(f_n - f_m)| dv(\mu),$$

thus we may assume that $v(\mu, S) < \infty$. By the first part of the proof of Theorem 11 it now follows that $N(f_n - f_m) \to 0$ and thus $f \in P$ and

$N(f_n - f) \to 0$. It remains to show $f \in P_S$. For each $n$, $f_n \in P_S$; hence there is a simple integrable function $h_n$ such that $N(f_n - h_n) < 1/n$. Then

$$N(h_n - h_m) \le 1/n + 1/m + N(f_n - f_m)$$

and thus $N(h_n - h_m) \to 0$. For each $x^*$ we have

$$|x^*(f_n - h_n)|_1 \le |x^*|N(f_n - h_n)$$

so that $x^*(f_n - h_n) \xrightarrow{\mu} 0$. By (1) it follows that $x^*h_n - \mu \to x^*f$ and Corollary 6 shows that $f \in P_S$.

Conversely, suppose $f \in P_S$ and $N(f_n - f) \to 0$. Since $f_n$, $f \in P$, the converse of Theorem 11 shows (1) and (2) hold. To show (3), let $\varepsilon > 0$. Choose $n_\varepsilon$ such that $N(f_n - f) < \varepsilon/2$ for $n \ge n_\varepsilon$. Since $f \in P_S$ there is by Lemma 12 a set $E_* \in \Sigma$ such that $v(\mu, E_*) < \infty$ and $\int_{S - E_*} |x^*f| dv(\mu) < \varepsilon/2$ for all $|x^*| \le 1$. Thus

$$\int_{S - E_*} |x^*f_n| dv(\mu) \le \int_{S-E_*} |x^*(f_n - f)| dv(\mu) + \int_{S - E_*} |x^*f| dv(\mu)$$

$$\le N(f_n - f) + \varepsilon/2 < \varepsilon$$

for all $n \ge n_\varepsilon$. For each $n < n_\varepsilon$, $f_{n_\varepsilon} \in P_S$; hence there is a set $E_{n\varepsilon} \in \Sigma$ such that $v(\mu, E_{n\varepsilon}) < \infty$ and $\int_{S-E_{n\varepsilon}} |x^*f_n| dv(\mu) < \varepsilon$. For $E_\varepsilon = E_* \bigcup (\bigcup_{n=1}^{n_\varepsilon - 1} E_{n_\varepsilon})$, we have $v(\mu, E_\varepsilon) < \infty$ and $\int_{S - E_\varepsilon} |x^*f_n| dv(\mu) < \varepsilon$ for all $n$.

Lemma 13. Let $\lambda(E) = \sum_{i=1}^{\infty} x_i \mu(EE_i)$ where $x_i \in \Xi$ and the series converges unconditionally for every $E \in \Sigma$. Then $\sum_i |x_i| v(\mu, EE_i)$ converges iff $v(\lambda, E) < \infty$.

Proof: For convenience assume $E = S$. Let $(A_n)_{n=1}^{k}$ be a disjoint sequence in $\Sigma$. Then

$$\begin{aligned}
\sum_n |\lambda(A_n)| &= \sum_n |\sum_i x_i \mu(A_n E_i)| \\
&\leq \sum_n \sum_i |x_i| v(\mu, A_n E_i) \\
&= \sum_i |x_i| \sum_n v(\mu, A_n E_i) \\
&\leq \sum_i |x_i| v(\mu, E_i);
\end{aligned}$$

hence

$$v(\lambda, S) \leq \sum_i |x_i| v(\mu, E_i).$$

Conversely, fix n and let $\varepsilon > 0$. For each $i = 1, \ldots, n$, choose $F_i \subset E_i$ such that

$$4|\mu(F_i)| > v(\mu, E_i) - \varepsilon/M$$

where $M = \sum_{i=1}^{n} |x_i|$. Then $(F_i)$ is disjoint and we have

$$\begin{aligned}
4v(\lambda, S) \geq 4\sum_{i=1}^{n} |\lambda(F_i)| &= 4\sum_{i=1}^{n} |\sum_j x_j \mu(F_i E_j)| \\
&= 4\sum_{i=1}^{n} |x_i \mu(F_i)| > \sum_{i=1}^{n} |x_i| (v(\mu, E_i) - \varepsilon/M \\
&= [\sum_{i=1}^{n} |x_i| v(\mu, E_i)] - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\sum_{i=1}^{n} |x_i| v(\mu, E_i) \leq 4v(\lambda, S).$$

This being true for all n, we have the desired inequality.

Lemma 14. Let $(\mu_n)$ be a sequence of measures from $\Sigma$ to $\Xi$ with uniformly bounded variation on S. If the limit $\lambda(E) = \lim_{n\to\infty} \mu_n(E)$ exists for all $E \in \Sigma$, then $v(\lambda, S) \leq \sup_n v(\mu_n, S)$.

Proof: Let $M = \sup_n v(\mu_n, S)$, $\varepsilon > 0$, and $(E_i)_{i=1}^{k}$ a finite disjoint sequence in $\Sigma$. For each i, choose an integer $n_i$ such that $|\lambda(E_i) - \mu_n(E_i)| < \varepsilon/k$ for $n \geq n_i$, and let $n_\varepsilon = \max_i n_i$. Then

$$\sum_i |\lambda(E_i)| \leq \varepsilon + \sum_i |\mu_{n_\varepsilon}(E_i)| \leq \varepsilon + v(\mu_{n_\varepsilon}, S) \leq \varepsilon + M.$$

Since $\varepsilon > 0$ is arbitrary it follows that $v(\lambda, S) \leq M$.

Theorem 15 (Pettis). If $\mu$ is countably additive; then $E \to \int_E f d\mu$ is countably additive.

In the remainder we assume $(S, \Sigma, \mu)$ is a measure space. In Theorem 16, Corollary 17, and Theorem 18, $(S, \Sigma, \mu)$ is assumed to be a $\sigma$-finite real measure space.

Theorem 16. Let f be a measurable function in $P(S, \Sigma, \mu, \Xi)$. Then $f \in L_1$ iff $v(\int f d\mu, S) < \infty$.

Proof: By Theorem 2 of Brooks [Br69] there is a bounded Bochner integrable function g, and a countably valued function

$h = \sum_{i=1}^{\infty} x_i 1_{E_i}$ where $(E_i)$ is a disjoint sequence, such that $f = g + h$, and $\int_E f d\mu = \int_E g d\mu + \sum_i x_i \mu(EE_i)$ where the series converges unconditionally for every $E \in \Sigma$. This theorem also states that $f \in L_1$ iff the series is absolutely convergent for each E. Let $\lambda(E)$ represent the series. By Lemma 14 the series is absolutely convergent iff $v(\lambda,E) < \infty$. Since $v(\int g d\mu,S) = \int_S |g| dv(\mu) < \infty$, it follows that $v(\lambda,S) < \infty$ iff $v(\int f d\mu,S) < \infty$.

Corollary 17. For measurable $f \in P$, $v(\int f d\mu,E) = \int_E |f| dv(\mu)$.

Proof: If $f \in L_1$, then the formula is known to hold. If f is not in $L_1$, then Theorem 16 shows that $v(\int f d\mu,S) = \infty$ which is consistent with the definition of the Bochner integral of the nonintegrable positive measurable function $|f|$.

Theorem 18. Let f be a measurable element of P. If $\lim_{n,m\to\infty} nv(\mu,\{s \in S: m < |f(s)| \leq n\}) = 0$, then $f \in L_1$.

Proof: Without loss of generality we may assume $\Sigma$ is complete. Let $\phi_n(E) = \int_{EE_n} f d\mu$ where $E_n = \{s \in S: |f(s)| \leq n\}$. Since $E_n \uparrow S$ and $\int f d\mu$ is countably additive, we have

$$\phi_n(E) = \int_{EE_n} f d\mu \to \int_E f d\mu \text{ for all } E \in \Sigma.$$

For all n and m with $n \geq m$ we have

$$v(\phi_n - \phi_m,S) = \int_{E_n - E_m} |f| dv(\mu) \leq nv(\mu,E_n - E_m) \to 0.$$

Thus $(\phi_n)$ is cauchy in $ca(S,\Sigma,\Xi)$; hence it is bounded. By Lemma 15, we have $v(\int f d\mu,S) \leq \sup_n v(\phi_n,S) < \infty$.

Theorem 19. Let $\Xi$ be a reflexive space which is separable or has a separable dual. Let $(f_n)$ be sequence in P such that $N(f_n - f_m) \to 0$. If $\sup_n |f_n(s)| < \infty$ for $\mu$-almost all $s \in S$, then there is a function $f \in P$ such that $N(f_n - f) \to 0$.

Proof: Let $(x_i^*)$ be a dense sequence in $\Xi^*$. For each $x^*$, $(x^* f_n)$ is cauchy in $L_1$ and therefore cauchy in measure. By Riesz's theorem there is a subsequence $(n_p)$ depending on $x^*$ such that $(x^* f_{n_p})$ converges $\mu$-almost everywhere. Using a diagonal process we obtain a subsequence $(n_k)$ such that $(x_i^* f_{n_k})$ converges almost everywhere for each i. Let E be the null set described and fix $s \in S - E$. Let $x_k^{*s} \in \Xi^{**}$ defined by $x_k^{*s} x^* = x^* f_{n_k}(s)$. By our construction the sequence $(x_k^{*s})_k$ converges on the fundamental set $(x_i^*)$ of $\Xi^*$ and by hypothesis we have $\sup_k |x_k^{*s} x^*| < \infty$. It follows from the Uniform Boundedness Principle that the limit $x^s x^* = \lim_k x_k^{*s} x^*$ exists for all $x^* \in \Xi$ and that $x^s \in \Xi^{**}$. Define $f(s) = x^s$. Then by hypothesis $f(s) \in \Xi$ and for each $x^*$ we have

$$x^* f_{n_k}(s) = x_k^{*s} x^* \to x^s x^* = x^* f(s).$$

Since $\mu$ is countably additive it follows that $x^* f_{n_k} - \mu \to x^* f$. Hence $x^* f_n - \mu \to x^* f$ and by Theorem 5 and Corollary 6 we have $f \in P$ and $N(f_n - f) \to 0$.

Corollary 20. If $\Xi$ is reflexive and $(f_n)$ is a sequence of measurable functions in P such that $N(f_n - f_m) \to 0$ and $\sup_n |f_n(s)| < \infty$ almost everywhere, then $f \in P$ and $N(f_n - f) \to 0$.

Proof: For each $f_n$ there is a sequence $(s_i^n)$ of simple functions such that $s_i^n \to f_n$ almost everywhere. Let $\Xi_0$ be the closed linear span of the set $\{s_i^n(S):i,n = 1,2,\ldots\}$. Then $(f_n)$ is a sequence in $P(S,\Sigma,\mu,\Xi_0)$ where $\Xi_0$ is reflexive and separable. By Theorem 19, there is a function $f \in P(S,\Sigma,\mu,\Xi_0)$ such that $N(f_n - f) \to 0$ in $P(S,\Sigma,\mu,\Xi_0)$. Clearly $f \in P(S,\Sigma,\mu,\Xi)$ and by the Hahn-Banach Theorem $N(f_n - f) \to 0$ in $P(S,\Sigma,\mu,\Xi)$.

Theorem 21. Assume $\Xi$ has the Radon-Nikodym Property. Let $(f_n)$ be a sequence in P such that $N(f_n - f_m) \to 0$ and suppose there is a disjoint sequence $(E_i)$ in $\Sigma$ such that $S = \bigcup_i E_i$ and the sequence $(f_n)$ is uniformly bounded in $L_1(S,\Sigma|_{E_i}, \mu|_{\Sigma \cap E_i},\Xi)$ for each i. Then there is a function $f \in P$ such that $N(f_n - f) \to 0$.

Proof: By lemma 3, $\lambda(E) = \lim_{n\to\infty}\int_E f_n d\mu$ exists for each $E \in \Sigma$. By the Vitali-Hahn-Saks Theorem it follows that $\lim_{\nu(\mu,E)\to 0}\lambda(E) = 0$. By Lemma 15, we have $\nu(\lambda,E_i) < \infty$ for all i. Since $\Xi$ has the Radon-Nikodym Property there is a function $f^i \in L_1(S,\Sigma|_{E_i}, \mu|_{\Sigma \cap E_i},\Xi)$ such that $\lambda(EE_i) = \int_{EE_i} f^i d\mu$ for every $E \in \Sigma$. Let $x^* \in \Xi^*$, $E \in \Sigma$. We have

$$(1) \qquad \int_{EE_i} x^* f^i d\mu = x^* \int_{EE_i} f^i d\mu = x^* \lambda(EE_i)$$

$$= x^* \lim_{n\to\infty} \int_{EE_i} f_n d\mu$$

$$= \lim_{n\to\infty} \int_{EE_i} x^* f_n d\mu.$$

Since

$$\int_{E_i} |x^*(f_n - f_m)| dv(\mu) \leq N(f_n - f_m) \to 0$$

and since $L_1$ is complete, there is a function $g_{x^*} \in L_1(E_i)$ such that

$$\int_{E_i} |x^* f_n - g_{x^*}| dv(\mu) \to 0.$$

Hence by (1) we have for each $E \in \Sigma$,

$$\int_{EE_i} x^* f^i d\mu = \lim_{n\to\infty} \int_{EE_i} x^* f_n d\mu = \int_{EE_i} g_{x^*} d\mu.$$

By Lemma III.6.8 of Dunford and Schwartz [Du58] it follows that $x^* f^i = g_{x^*}$ $\mu$-almost everywhere on $E_i$. Then

$$x^* f_n - \mu \to x^* f^i$$

on $E_i$. Define f on S by $f(s) = f^i(s)$ for $s \in E_i$, and let $x^* \in \Xi^*$, and $E_n^i = \{s \in E_i : |x^*(f_n(s) - f^i(s))| > \delta\}$. Since $\mu$ is countably additive we may assume $E_n^i \in \Sigma$. For each i there is an integer $n_i$ such that $v(\mu, E_n^i) < \varepsilon/2^{i+1}$ for $n \geq n_i$. We may choose $(n_i)$ such that $n_{i+1} > n_i$ for every i. Then for k and $n \geq n_k$ we have

$$v(\mu, \{s \in S : |x^*(f_n(s) - f(s))| > \delta\})$$

$$= \sum_i v(\mu, E_n^i) = \sum_{i=1}^{k} v(\mu, E_n^i) + \sum_{i=k+1}^{\infty} v(\mu, E_n^i)$$

$$< \varepsilon/2 + \sum_{i=k+1}^{\infty} v(\mu, E_n^i).$$

Since $\mu$ is strongly additive,

$$\sum_{i=k+1}^{\infty} v(\mu, E_n^i) \to 0 \text{ as } k \to \infty.$$

Thus for $k_0$ such that $\sum_{i=k+1}^{\infty} v(\mu, E_n^i) < \varepsilon/2$, $n \geq n_{k_0}$ implies $v(\mu, \{s \in S: |x^*(f_n(s) - f(s))| > \delta\}) < \varepsilon$; hence $x^* f_n - \mu \to x^* f$ for all $x^*$. By Theorem 5 and Corollary 6, it follows that $f \in P$ and $N(f_n - f) \to 0$.

## II.2 Integration With Respect to a Vector Valued Measure

Let $(X,\tau)$ be a complete Hausdorff locally convex space with neighborhood basis $\Psi$. Without loss of generality we may assume each $U \in \Psi$ is balanced and convex. Thus for each $U \in \Psi$ the gauge p of U satisfies $p(\alpha x) = |\alpha| p(x)$ for $x \in X$, $\alpha$ scalar.

Lemma 1. For each continuous linear functional $x^*$ on X there is a continuous seminom p such that $|x^* x| \leq p(x)$ for every $x \in X$. In this case we write $x^* \leq p$.

Proof: Let $x^* \in X^*$, $U = \{x: |x^* x| < 1\}$. Then $U \in \Psi$. Let p be the gauge of U, and fix $x_0 \in X$. If $p(x_0)$ is nonzero, then we have $p(x_0/p(x_0)) \leq 1$; hence $x_0/p(x_0) \in \overline{U}$ and consequently

$$|x^* x_0| = p(x_0)|x^*(x_0/p(x_0))| \leq p(x_0).$$

Suppose $p(x_0) = 0$. Then $x_0 \in \alpha U$ for every $\alpha > 0$ or $x_0/\alpha \in U$; hence $|x^*(x_0/\alpha)| < 1$ and we have $|x^* x_0| < \alpha$. It follows that $|x^* x_0| = 0$ so that $|x^* x_0| \leq p(x_0)$ in this case also.

Let S be a set with an algebra $\Sigma$ of subsets and let $\mu:\Sigma \to X$ be an additive function. For each continuous seminorm on p on X define $|\mu|_p(E) = \sup_{x^* \leq p} v(x^*\mu,E)$. Clearly $|\mu|_p(E) \leq 4\sup_{F \subset E, F \in \Sigma} p(\mu(F))$. $|\mu|^*_p(A)$ denotes the extension $\inf_{A \subset E, E \in \Sigma} |\mu|_p(E)$ of $|\mu|_p$ to P(S).

Lemma 2. $p(\mu(E)) \leq |\mu|_p(E)$.

Proof: Let $X_0$ be the one dimensional subspace $\{\alpha\mu(E):\alpha \text{ scalar}\}$. For $x = \alpha\mu(E) \in X_0$ define $x^*x = \alpha p(\mu(E))$. Then $|x^*x| = |\alpha| p(\mu(E)) = p(x)$; hence $x^* \leq p$ on $X_0$. By the Hahn-Banach theorem $x^*$ can be extended to X in such a way that $x^*x \leq p(x)$ for all $x \in X$. Then $x^* \leq p$ on X and thus

$$p(\mu(E)) = |x^*\mu(E)| \leq v(x^*\mu,E) \leq |\mu|_p(E).$$

Theorem 3. If $\mu$ is weakly countably additive, then it is strongly countably additive (see Grothendieck [Gr53]).

Theorem 3b. If $\mu$ is countably additive and $(E_n)$ is a monotone sequence in $\Sigma$ with limit $E \in \Sigma$, then $|\mu|_p(E_n) \to |\mu|_p(E)$. (See Lewis [Le70].)

Note: Lemma 2 also follows from the fact that $|\mu|_p(E) = \sup(\Sigma_i \alpha_i \mu(EE_i))$ where the supremum is taken over all finite sets $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ of scalars with $|\alpha_i| \leq 1$ and all finite sets $\{E_1,E_2,\ldots,E_n\}$ of disjoint sets in $\Sigma$. (See Proposition 1.11(a) of Diestel and Uhl [Di77] and Corollary 1.7.2 of Edwards [Ed65] from which the equation $p(x) = \sup\{|x^*x|:p(x^*) \leq 1\}$ for $p(x^*) = \sup_{p(x)<1} |x^*x|$ follows.)

A scalar valued function f defined on S is $\mu$-measureable if for every continuous seminorm p and $\varepsilon,\delta>0$ there is a simple function $\phi$

such that $|\mu|^*_p(\{s:|f(s) - \phi(s)| > \varepsilon\}) < \delta$. Note that in this case there is for each such p a sequence $(\phi_n)$ of simple functions such that $|\mu|^*_p(\{s:|f(s) - \phi_n(s)| > \varepsilon\}) \to 0$ for all $\varepsilon > 0$. If such a sequence can be chosen independently of the seminorms p, then f will be called sequentially measurable. If X is metrizable or if $\mu$ has a control measure, then every measurable function is sequentially measurable. A sequence $(f_n)$ converges in $\mu$-measure to f if $|\mu|^*_p(\{s:|f(s) - f_n(s)| > \varepsilon\}) \to 0$ (equivalently, $f_n \to f$ in $p(\mu)$-measure for every $\varepsilon > 0$). A scalar valued function f defined on S is $\mu$-integrable, or simply integrable, if $f \in L_1(S,\Sigma,x^*\mu)$ for every $x^* \in X^*$ and if for every $E \in \Sigma$ there is an element $x_E \in X$ such that $x^*x_E = \int_E f d(x^*\mu)$ for every $x^* \in X^*$. In this case the integral of f over E is given by $\int_E f d\mu = x_E$.

Lemma 4. The set function $E \to \int_E f d\mu$ is strongly additive and $\mu$-continuous.

Proof: Let $\lambda(E) = \int_E f d\mu$. For every $x^*\lambda = \int f d(x^*\mu)$ is a measure of bounded variation and is therefore strongly additive. By the Orlicz-Pettis theorem [Ro69] it follows that the integral is strongly additive. Since $|x^*\lambda(E)| \leq p(\lambda(E))$ for all $x^* \leq p$ it follows that $\{x^*\lambda : x^* \leq p\}$ is uniformly strongly additive. Since $v(x^*\lambda) = \int |f| dv(x^*\mu) << v(x^*\mu) << x^*\mu \leq p(\mu)$ for every $x^* \leq p$ if follows from Theorem 1 of Brooks [Br73] that $\{x^*\lambda : x^* \leq p\}$ is uniformly $p(\mu)$-continuous; hence $|\lambda|_p$ is $p(\mu)$-continuous.

Corollary 5. For f integrable, $\left|\int f d\mu\right| p(S) < \infty$ for every continuous p.

Proof: Since $\int f d\mu$ is strongly additive it is bounded. (See Rickart [Ri44].)

Lemma 6. For f bounded and integrable we have $\left|\int f d\mu\right|_p(E) \leq |f| \|\mu\|_p(E)$. Consequently one has $p(\int_E f d\mu) \leq |f| \|\mu\|_p(E)$.

Proof:

$$\left|\int f d\mu\right|_p(E) = \sup_{x^* \leq p} v(\int f d\mu, E)$$

$$= \sup_{x^* \leq p} \int_E |f| dv(x^*\mu)$$

$$\leq |f| \sup_{x^* \leq p} v(x^*\mu, E) = |f| \|\mu\|_p(E).$$

Lemma 7. The map $f \to \int_E f d\mu$ is linear in f.

Proof: Let f, g be integrable, $\alpha$ scalar. For every $x^* \in X^*$ we have $\alpha f + g \in L_1(S, \Sigma, x^*\mu)$; hence

$$x^*(\alpha \int_E f d\mu + \int_E g d\mu) = \alpha \int_E f d(x^*\mu) + \int_E g d(x^*\mu) = \int_E (\alpha f + g) d(x^*\mu)$$

and it follows that $\alpha f + g$ is integrable and that $\int_E (\alpha f + g) d\mu = \alpha \int_E f d\mu + \int_E g d\mu$.

Let I(X) denote the space of $\mu$-integrable functions. For each continuous seminorm p on X and $f \in I(X)$ define $p(f) = \left|\int f d\mu\right|_p(S)$. It is easy to verify that p can then be regarded as a seminorm on I(X) and that I(X) is therefore a locally convex space. If $(f_n)$ and f are $\mu$-integrable functions such that $p(f_n - f) \to 0$ for each p, then $(f_n)$ is said to converge to f in I(X).

Note that if $f_n \to f$ in $I(X)$, then for all $E \in \Sigma$ we have

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

Theorem 8. Let $f$ be a $\mu$-measurable $\mu$-integrable function where $\mu$ is bounded. Then for every continuous $p$ and $\varepsilon > 0$ there is a simple function $f_\varepsilon$ such that $p(f - f_\varepsilon) < \varepsilon$. If $\mu$ has a control measure, then there is a sequence $(f_n)$ of simple functions such that $p(f - f_n) \to 0$ for every $p$.

Proof: Let $p$ and $\varepsilon$ be given and let $(\phi_n)$ be a sequence of simple functions such that $|\mu|^*_p(\{s: |f(s) - \phi_n(s)| > \varepsilon\}) \to 0$ for all $\varepsilon > 0$. For each positive integer $k$ choose an integer $n_k$ and a set $E_k \in \Sigma$ such that $|\mu|_p(E_k) < 1/k$ and $|f(s) - \phi_{n_k}(s)| \leq 1/k$ for $s \in S - E_k$. Let $f_k(s) = \phi_{n_k}$ for $s \in S - E_k$ and $f_k(s) = 0$ for $s \in E_k$. Then $f_k$ is a simple function and for each $x^* \leq p$,

$$\begin{aligned}
\int_S |f - f_k| dv(x^*\mu) &= \int_{S-E_k} |f - f_k| dv(x^*\mu) + \int_{E_k} |f - f_k| dv(x^*\mu) \\
&= \int_{S-E_k} |f - \phi_{n_k}| dv(x^*\mu) + \int_{E_k} |f| dv(x^*\mu) \\
&\leq v(x^*\mu, S)/k + 4\sup_{F \in \Sigma, F \subset E_k} \left| \int_F f d(x^*\mu) \right| \\
&\leq v(x^*\mu, S)/k + 4\sup_{F \in \Sigma, F \subset E_k} p\left(\int_F f d\mu\right);
\end{aligned}$$

hence

$$p(f - f_k) \leq |\mu|_p(S)/k + 4\sup_{F\in\Sigma,\ F\subset E_k} p(\int_F f d\mu).$$

Since $\lim_{|\mu|_p(E)\to 0} p(\int_E f d\mu) = 0$ there is an integer $k_\varepsilon$ such that $p(f - f_{k_\varepsilon}) < \varepsilon$. Note that

$$p(\int_E (f - f_{k_\varepsilon}) d\mu) \leq |\int (f - f_{k_\varepsilon}) d\mu|_p(E) \leq |\int (f - f_{k_\varepsilon}) d\mu|_p(S) = p(f - f_{k_\varepsilon});$$

and hence

$$p(\int_E f d\mu - \int_E f_{k_\varepsilon} d\mu) < \varepsilon \text{ for all } E \in \Sigma.$$

In the case $\mu$ has a control measure then f is sequentially measurable and the sequences $(f_k)$ and $(E_k)$ can be chosen independently of the semi-norms p hence $p(f - f_k) \to 0$ for every p. Note that $p(\int_E f d\mu - \int_E f_n d\mu) \to 0$ for every p and $E \in \Sigma$ thus $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$ for every $E \in \Sigma$.

$I_s(X)$ will denote the subspace of $I(X)$ consisting of those functions f for which there is a sequence $(f_n)$ of simple functions such that $f_n \to f$ in $I(X)$.

Corollary 9. If X is a Banach space and $\mu$ is countably additive, then the measurable integrable functions are the Bartle-Dunford-Schwarz integrable functions.

Proof: Since $\mu$ is countably additive it is bounded and has a control measure $\lambda < \infty$. If f is measurable and integrable, then there is by Theorem 8 a sequence $(f_n)$ of simple functions converging in measure to f (f is sequentially measurable) such that $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$. Since $\lambda \ll \mu$ we have $f_n - \lambda \to f$ and by Corollary III.13(a) of Dunford and Schwartz [Du58] there is a subsequence $(f_{n_k})$ converging $\lambda$-a.e.

and therefore $\mu$-a.e. to f. Thus f is Bartle-Dunford-Schwarz integrable with defining sequence $(f_{n_k})$. The converse follows from Theorem III.8(f) of Dunford and Schwartz [Du58].

Lemma 10. If $\mu$ is bounded on a $\sigma$-algebra $\Sigma$ and $(f_n)$ is a sequence of integrable functions converging in $x^*\mu$-measure for every $x^*$ to a function f and if $\lim_n \int_E f_n d\mu$ exists for each $E \in \Sigma$, then f is integrable and $\int_E f d\mu = \lim_n \int_E f_n d\mu$.

Proof: Let $x^* \in X^*$. Since $\lim_n \int_E f_n d\mu$ exists it follows that $\lim_n \int_E f_n d(x^*\mu)$ exists for every E; hence $\lim_{v(x^*\mu,E)\to 0} \int_E f_n d(x^*\mu) = 0$ uniformly in n by Theorem 3 of Brooks and Jewett [Br70]. Since $f_n \to f$ in $x^*\mu$-measure it follows that $f \in L_1(S,\Sigma,x^*\mu)$ and $f_n \to f$ in $L_1$. Then

$$x^* \lim_n \int_E f_n d\mu = \lim_n \int_E f_n d(x^*\mu) = \int_E f d(x^*\mu)$$

so that by definition we have f integrable with

$$\int_E f d\mu = \lim_n \int_E f_n d\mu.$$

Theorem 10a. Let $(f_n)$ be a sequence of integrable functions and $f:S \to X$. If $\lim_{n,m\to\infty} p(f_n - f_m) \to 0$ for every p and $f_n$ converges in $x^*\mu$-measure to f for all $x^*$, then f is integrable and $p(f_n - f) \to 0$ for every p (hence $f \in I_s$).

Proof: Let $x^* \in X^*$. By Lemma 2 there is a continuous seminorm p such that $x^* \leq p$; hence

$$\int_S |f_n - f_m| dv(x^*\mu) \leq p(f_n - f_m) \to 0.$$

Since $f_n - x^*\mu \to f$ it follows that $f \in L_1(x^*\mu)$ and that $f_n \to f$ in $L_1(x^*\mu)$. Let $E \in \Sigma$, p a continuous seminorm. Since $p(\int_E f_n d\mu - \int_E f_m d\mu) \leq p(f_n - f_m)$ it follows that the limit $\lim_{n\to\infty} \int_E f_n d\mu$ exists. Denoting this limit by $x_E$ we have $x^* x_E = \lim_{n\to\infty} \int_E f_n d(x^*\mu) = \int_E f d(x^*\mu)$ thus f is integrable and

$$\int_E f d\mu = x_E = \lim_{n\to\infty} \int_E f_n d\mu.$$

Fix p and let $\varepsilon > 0$. Choose $n_\varepsilon$ such that $p(f_n - f_m) < \varepsilon$ for $n,m \geq n_\varepsilon$. Then for $n \geq n_\varepsilon$ and $x^* \leq p$ we have

$$\int_S |f_n - f| dv(x^*\mu) = \lim_{m\to\infty} \int_S |f_n - f_m| dv(x^*\mu) \leq \lim_{n\to\infty} p(f_n - f_m) < \varepsilon$$

thus $p(f_n - f) \leq \varepsilon$ for $n \geq n_\varepsilon$.

Corollary 11. $f \in I_s(X)$ iff there is a sequence $(f_n)$ of simple functions such that $p(f_n - f_m) \to 0$ for each p and $f_n \to f$ in $x^*\mu$-measure for each $x^* \in X^*$.

Theorem 12 (Vitali Convergence Theorem for bounded $\mu$). Assume $\mu$ is bounded. Let $(f_n)$ be a sequence of $\mu$-integrable functions and $f:S \to X$. Then f is integrable and $f_n \to f$ in $I(X)$ if

(1) $f_n \to f$ in $p(\mu)$-measure for each p, and

(2) $\lim_{\mu(E)\to 0} \int_E f_n d\mu = 0$ uniformly in n.

Proof: Fix $p$ and $\varepsilon > 0$. Let $\delta > 0$ such that $p(\int_E f_n d\mu) < \varepsilon$ for $p(\mu(E)) < \delta$. Choose $n_\delta$ such that for $n,m \geq n_\delta$ there are sets $F_{nm} \in \Sigma$ such that $p(\mu(F_{nm})) < \delta$ and $|f_n(s) - f_m(s)| \leq \varepsilon$ for $s \in S - F_{nm}$. Then for $E \in \Sigma$ and $n,m \geq n_\delta$ we have

$$p(\int_E (f_n - f_m)d\mu) \leq p(\int_{E-F_{nm}} (f_n - f_m)d\mu) + p(\int_{F_{nm}} (f_n - f_m)d\mu)$$

$$\leq \varepsilon |\mu|_p(S) + 2\varepsilon;$$

hence

$$p(f_n - f_m) = |\int (f_n - f_m)d\mu|_p(S) \leq 4\sup_{E\in\Sigma} p(\int_E (f_n - f_n)d\mu) \leq 4\varepsilon(|\mu|_p(S) + 2).$$

It follows that $p(f_n - f_m) \to 0$ for each $p$. Since $f_n \to f$ in $p(\mu)$-measure the conclusion now follows from Theorem 10 and Lemma 1.

Corollary 13 (Dominated Convergence Theorem for bounded $\mu$). Let $(f_n)$ be a sequence of integrable functions such that $f_n \to f$ in $p(\mu)$-measure and $|f_n(s)| \leq g(s)$ for all $s$ in the compliment of a $\mu$-null set where $g$ is integrable. Then $f$ is integrable and $f_n \to f$ in $I(X)$.

Proof: Since

$$p(\int_E f_n d\mu) \leq |\int f_n d\mu|_p(E) = \sup_{x^* \leq p} \int_E |f_n| dv(x^*\mu)$$

$$\leq \sup_{x^* \leq p} \int_E g dv(x^*\mu)$$

$$= |\int g d\mu|_p(E),$$

it follows that $\lim_{\mu(E)\to 0} \int_E f_n d\mu = 0$ uniformly in $n$ and thus the result follows from the Vitali Convergence Theorem.

Lemma 14. Let $f \in I_s(X)$. For each $p$ and $\varepsilon > 0$ there is a set $E_{p,\varepsilon} \in \Sigma$ such that $|\mu|_p(E_{p,\varepsilon}) < \infty$, and $|\int f d\mu|_p(S - E_{p,\varepsilon}) < \varepsilon$.

Proof: Let $p, \varepsilon > 0$ be given. Let $(f_n)$ be a sequence of simple functions such that $p(f_n - f) \to 0$ for each $p$. Choose $n_{p,\varepsilon}$ such that $p(f_n - f) < \varepsilon$ for $n \geq n_{p,\varepsilon}$. Let $E_{p,\varepsilon}$ be the union of the sets $f_{n_{p,\varepsilon}}^{-1}(x)$ for $x \in X - \{0\}$. Since $f_{n_{p,\varepsilon}}$ is simple, $E_{p,\varepsilon} \in \Sigma$ and $|\mu|_p(E_{p,\varepsilon}) < \infty$. We have

$$|\int f d\mu|_p(S - E_{p,\varepsilon}) = \sup_{x^*} \leq_p \int_{S-E_{p,\varepsilon}} |f| dv(x^*\mu)$$

$$\leq \sup_{x^*} \leq_p \int_{S-E_{p,\varepsilon}} |f - f_{n_{p,\varepsilon}}| dv(x^*\mu) + \sup_{x^*} \leq_p \int_{S-E_{p,\varepsilon}} |f_{n_{p,\varepsilon}}| dv(x^*\mu)$$

$$= \sup_{x^*} \leq_p \int_{S-E_{p,\varepsilon}} |f - f_{n_{p,\varepsilon}}| dv(x^*\mu)$$

$$\leq p(f - f_{n_{p,\varepsilon}}) < \varepsilon.$$

Theorem 15 (Vitali Convergence Theorem for $I_s(X)$). Let $(f_n)$ be a sequence of integrable functions converging in $p(\mu)$-measure for each $p$, to a function $f$ such that

(1) $\lim_{\mu(E)\to 0} \int_E f_n d\mu = 0$ uniformly for $n = 1, 2, \ldots$, and

(2) for every $p$ and $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ such that $|\mu|_p(E) < \infty$ and $p(\int_F f_n d\mu) < \varepsilon$ for all $n$ and $F \in \Sigma$ with $F \subset S - E_\varepsilon$.

Then $f$ is integrable and $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$ for $E \in \Sigma$.

Proof: Let $p, \varepsilon > 0$ be given and choose $E_{p,\varepsilon}$ by (2), and $\delta > 0$ by (1) such that $p(\int_E f_n d\mu) < \varepsilon$ for $p(\mu(E)) < \delta$. Since $(f_n - f_m)_{n,m}$ converges

in $p(\mu)$-measure to 0 there is a number $n_p$ such that for $n,m \geq n_p$ there is a set $F_{n,m} \in \Sigma$ such that $p(\mu(F_{n,m})) < \delta$ and $|f_n(s) - f_m(s)| \leq \varepsilon$ for $s \in S - F_{n,m}$. For $E \in \Sigma$ we have

$$p\left(\int_E (f_n - f_m)d\mu\right)$$

$$\leq p\left(\int_{E-(E_{p,\varepsilon} \cup F_{n,m})} (f_n - f_m)d\mu\right) + p\left(\int_{E_{p,\varepsilon}-F_{n,m}} (f_n - f_m)d\mu\right)$$

$$+ p\left(\int_{F_{n,m}} (f_n - f_m)d\mu\right)$$

$$\leq 2\varepsilon + \varepsilon|\mu|_p(E_{p,\varepsilon}) + 2\varepsilon,$$

hence the sequence $\left(\int_E f_n d\mu\right)$ is cauchy. Let $x_E = \lim_{n\to\infty}\int_E f d\mu$, $x^* \in X^*$. Then $\int_E f_n d(x^*\mu) \to x^* x_E$ for each $E \in \Sigma$. By the Vitali Han Saks theorem it follows that $\lim_{v(x^*\mu,E)\to 0}\int_E |f_n| dv(x^*\mu) = 0$ uniformly in $n$. By Lemma 2 there is a seminorm $p$ such that $x^* \leq p$. For $\varepsilon > 0$ there is by (2) a set $E_{p,\varepsilon} \in \Sigma$ such that $|\mu|_p(E_{p,\varepsilon}) < \infty$ and $p\left(\int_F f_n d\mu\right) < \varepsilon$ for all $n$ and $F \subset S - E_{p,\varepsilon}$. Then we have $v(x^*\mu, E_{p,\varepsilon}) \leq |\mu|_p(E_{p,\varepsilon}) < \infty$, and

$$\int_{S-E_{p,\varepsilon}} |f_n| dv(x^*\mu) \leq \left|\int f_n d\mu\right|_p (S - E_{p,\varepsilon})$$

$$\leq 4 \sup_{F \subset S-E_{p,\varepsilon}} p\left(\int_F f_n d\mu\right) \leq 4\varepsilon.$$

Since $(f_n)$ converges in $p(\mu)$-measure to $f$ and $x^* \leq p$ it follows that $(f_n)$ converges in $x^*\mu$-measure to $f$. Now by the classical Vitali Convergence Theorem it follows that $f \in L_1(x^*\mu)$ and $f_n \to f$ in

$L_1(x^*\mu)$. Finally, since $x^*x_E = \lim_{n\to\infty}\int_E f_n d(x^*\mu) = \int_E f d(x^*\mu)$ it follows that f is $\mu$-integrable and $\int_E f d\mu = x_E = \lim_{n\to\infty}\int_E f_n d\mu$ for all $E \in \Sigma$.

Lemma 16. If $\mu$ is countably additive and bounded on a $\sigma$-algebra, then a sequence of functions convergent almost everywhere is convergent in $|\mu|_p$ measure.

Proof: The classical proof of Egoroff's theorem remains valid in this context.

Lemma 17. $|\mu|_p(S) = \sup_{|f|\le 1} p(\int_S f d\mu)$ for $f \in B(S,\Sigma)$. In fact the supremum may be taken just over simple functions.

Proof: First, we have

$$
\begin{aligned}
|\mu|_p(S) &= \sup_{x^*\le p} v(x^*\mu, S) \\
&= \sup_{x^*\le p} \sup_{(E_i)} \Sigma |x^*\mu(E_i)| \\
&= \sup_{x^*\le p} \sup_{(E_i)} \Sigma x^*(\operatorname{sign}(x^*\mu(E_i))\mu(E_i)) \\
&= \sup_{x^*\le p} \sup_{(E_i)} x^*(\Sigma \operatorname{sign}(x^*\mu(E_i))\mu(E_i)) \\
&\le \sup_{(E_i)} p(\Sigma \operatorname{sign}(x^*\mu(E_i))\mu(E_i)) \\
&\le \sup_{|f|\le 1} p(\int_S f d\mu),\ f \text{ simple} \\
&\le \sup_{|f|\le 1} p(\int_S f d\mu),\ f \in B(S,\Sigma),
\end{aligned}
$$

where $(E_i)$ is an arbitrary finite disjoint sequence in $\Sigma$. On the

other hand, by a corollary to the Hahn-Banach theorem (see [Ed65] corollary 1.7.2), for every $f \in B(S,\Sigma)$ with $|f| \leq 1$ we have

$$\begin{aligned} p(\textstyle\int_E f d\mu) &= \sup_{x^* \leq p} |x^* \textstyle\int_E f d\mu| \\ &= \sup_{x^* \leq p} |\textstyle\int_E f d(x^*\mu)| \\ &\leq \sup_{x^* \leq p} v(x^*\mu, E) \\ &= |\mu|_p(E). \end{aligned}$$

Lemma 18. If $\mu$ is bounded, then every element of $B(S,\Sigma)$ is $\mu$-integrable. In fact $B(S,\Sigma) \subset I_s$ in this case.

Proof: Let $f \in B(S,\Sigma)$ and let $(f_n)$ be a sequence of simple functions such that $f_n \to f$ in $B(S,\Sigma)$. Then $f_n \to f$ in $p(\mu)$-measure for every $p$ since convergence in $B(S,\Sigma)$ is uniform. By Lemma 6 we have $p(f_n - f_m) \leq |f_n - f_m| |\mu|_p(S) \to 0$ for every $p$ hence $f$ is $\mu$-integrable and $f_n \to f$ in $I(X)$ by Theorem 10.

Note: If $\mu:\Sigma \to X$ is a bounded additive function and $Tf = \int_S f d\mu$, then by Lemmas 6 and 7, $T$ is a bounded linear operator on $B(S,\Sigma)$.

Theorem 9. For each bounded linear operator $T:B(S,\Sigma) \to X$ there is a unique bounded additive function $\mu:\Sigma \to X$ such that $Tf = \int_S f d\mu$.

Proof: For $E \in \Sigma$ let $\mu(E) = T(1_E)$. Then $\mu$ is additive and bounded on $\Sigma$ and $T(1_E) = \int_S 1_E d\mu$. For $f$ simple it follows from the linearity of $T$ that $Tf = \int_S f d\mu$. Since the simple functions are dense in $B(S,\Sigma)$ and

the continuous functions $f \to Tf$ and $f \to \int_S f d\mu$ agree on a dense set it follows that $Tf = \int_S f d\mu$ for all $f \in B(S,\Sigma)$. For uniqueness, suppose $\nu$ also represents $T$. Then $\int_S f d\mu = \int_S f d\nu$ for all $f \in B(S,\Sigma)$, in particular for $f = 1_E$ we have

$$\mu(E) = \int_E 1 d\mu = \int_S 1_E d\mu = \int_S 1_E d\nu = \int_E 1 d\nu = \nu(E)$$

so $\nu = \mu$.

Theorem 20. Let $T$ be a linear operator on $C(S)$ with values in $X$ where $S$ is a normal space. Then there is a unique additive function $\mu$ defined on the Borel subsets of $S$ with values in $X^{**}$ such that

(1) $\mu x^* \in rba(S)$ for every $x^*$,

(2) $T^*: x^* \to \mu x^*$ is continuous for the weak $^*$ topologies, and

(3) $x^* Tf = \int_S f d(\mu x^*)$ for each $x^*$ and $f \in C(S)$.

If $S$ is a compact Hausdorff space, then

(1′) $\mu x^* \in rca(S)$ for every $x^*$.

If $T$ is bounded ($p(T) = \sup\{p(Tf): |f| \leq 1\} < \infty$ for all $p$), then $\mu$ is bounded with

(4) $|\mu|_p(S) = p(T)$.

If $T$ is weakly compact, then $\mu(E) \in X$ for all $E \in \Sigma$. Thus each $f \in C(S)$ is $\mu$-integrable ($\mu$ and $f$ bounded) and we have

(3′) $Tf = \int_S f d(\mu)$ on $C(S)$ ($x^* Tf = \int_S f d(\mu x^*)$ for each $x^*$).

If (1′) holds and $\mu$ is $X$ valued, then $\mu$ is countably additive by the Orlicz-Pettis theorem [Ro58].

Conversely, given $\mu: \Sigma \to X^{**}$ satisfying (1) or (1′), and (2), there is a unique linear operator $T: C(S) \to X$ satisfying (3). If $\mu$ is

bounded, then T is bounded and satisfies (4). If $\mu$ is X valued, then T satisfies (3′).

Proof: For any Borel set B let $\lambda_B^*$ be the functional on rba(S) defined by $\lambda_B^*\lambda = \lambda(B)$. Theorem IV.6.2 of Dunford and Schwartz [Du58] we have $\lambda_B^* \in \text{rba}^*(S) = C^{**}(S)$. Let $\mu(B) = T^{**}\lambda_B^*$. Since $\mu(B)x^* = T^{**}\lambda_B^* x^* = \lambda_B^* T^* x^* = T^* x^*(B)$ we see that $\mu$ satisfies (1) and (2) and we have (3) $x^*Tf = T^*x^*f = \int_S f d(\mu x^*)$. Since $\mu x^*$ is regular and X is Hausdorff it follows that $\mu$ is uniquely determined by (3). If S is compact and Hausdorff, then rba(S) = rca(S) and we have (3′). For $|\alpha_i| \leq 1$, $E_i \in \Sigma$, and $p(x^*) = \sup\{|x^*x| : p(x) \leq 1\}$ (see the note following Theorem 3b) we have

$$\begin{aligned} |\mu|_p(S) &= \sup_{|f|\leq 1} p\left(\int_S f d\mu\right) \\ &= \sup_{|f|\leq 1} \sup_{x^*\leq p} \left|\int_S f d(\mu x^*)\right| \\ &= \sup_{|f|\leq 1} \sup_{x^*\leq p} |x^*Tf| \\ &= \sup_{|f|\leq 1} p(Tf) \\ &= p(T). \end{aligned}$$

Thus $\mu$ is bounded if and only if T is bounded. The last statement follows from Theorem 9.3.1 of Edwards [Ed65] and the definition of $\mu$. Conversely, if (1) or (1′) and (2) hold for an additive $\mu$, then for fixed $f \in C(S)$ the map $x^* \to \int_S f d(\mu x^*)$ is continuous for the X topology on $X^*$. By Corollary V.3.11 of Dunford and Schwartz [Du58] there is an element $x_f \in X$ such that $\int_S f d(\mu x^*) = x^* x_f$. Then

for $Tf = x_f$, T is linear and satisfies (3) or (3′). Since X is Hausdorff, (3) (or (3′)) uniquely determines T. As already shown, T is bounded if $\mu$ is bounded and we have (4).

Corollary 21. If $T:C(S) \to X$ is a weakly compact bounded linear operator, then T transforms weak cauchy sequences into strongly convergent sequences.

Proof: Let $\mu:\Sigma \to X$ be a bounded countably additive vector measure such that $Tf = \int_S f d\mu$ on C(S). By Lemma 18 every element of $B(S,\Sigma)$ is $\mu$-integrable. Let $(f_n)$ be a weak cauchy sequence in C(S). First note $(f_n)$ is bounded since it is weakly bounded. Since the evaluation functionls on C(S) are continuous it follows that $(f_n(s))$ is cauchy for every $s \in S$ thus $f(s) = \lim_n f_n(s)$ exists. Since a bounded function g is in $B(S,\Sigma)$ if and only if $g^{-1}(H) \in \Sigma$ for every open set H, it follows from Theorem 20A of Halmos [Ha74] that f is in $B(S,\Sigma)$. By Lemma 16, $f_n \to f$ in $p(\mu)$-measure for each p, thus by Theorem 13, f is integrable and $p(f_n - f) \to 0$, in particular $\int_E f_n d\mu \to \int_E f d\mu$ for each E.

Theorem 22. Let $\psi \in ba(S,\Sigma)$ and let T be a bounded linear operator from $L_1(S,\Sigma,\psi)$ into X. Define $\mu$ from $\Sigma$ into X by $\mu(E) = T(1_E)$. Then for each $f \in L_1(S,\Sigma,\psi)$ f is $\mu$-integrable and $Tf = \int_S f d\mu$.

Proof: Let $f \in L_1$ and let $(f_n)$ be a sequence of simple functions converging to f in $L_1$. Fix $p, \varepsilon > 0$. Since T is bounded

$(p(T) = \sup_{|f|\le 1} p(Tf) < \infty$ for all $p)$, there is a number $\delta>0$ such that $|f| < \delta$ implies $p(Tf) < \varepsilon$. Since $\lim_{\nu(\psi,E)\to 0}\int_E |f_n| d\nu(\psi) = 0$ uniformly in $n$, there is a number $\xi > 0$ such that $\int_E |f_n| d\nu(\psi) < \delta$ for all $n$ and $E$ with $\nu(\psi,E) < \xi$. Since $|f_n 1_E|_1 = \int_S |f_n| 1_E d\nu(\psi) = \int_E |f_n| d\nu(\psi)$, we have $\nu(\psi,E) < \xi$ implies $|f_n 1_E|_1 < \delta$ for all $n$ and thus $p(T(f_n 1_E)) < \varepsilon$ for all $n$. Since $T(f_n 1E) = \int_E f_n d\mu$ we have proved that $\lim_{\nu(\psi,E)\to 0} p(\int_E f_n d\mu) = 0$ uniformly in $n$. By the classical Vitali Convergence theorem, convergence in $L_1$ of $f_n$ to $f$ implies convergence in $\psi$-measure. Thus for each positive integer $k$ we may choose $n_k$ such that for $n,m \ge n_k$ there is a set $F_{nk} \in \Sigma$ with $\nu(\psi,F_{nk}) < \xi$ and $|f_n(s) - f_m(s)| < \varepsilon$ for $s \in S - F_{nm}$. Then

$$
\begin{aligned}
p(\textstyle\int_E (f_n - f_m)) d\mu &\le p(\int_{E-F_{nm}} (f_n - f_m)) d\mu + p(\int_{EF_{nm}} (f_n - f_m) d\mu) \\
&\le \varepsilon |\mu|_p(S) + p(\textstyle\int_{EF_{nm}} f_n d\mu) + p(\int_{EF_{nm}} f_m d\mu) \\
&\le \varepsilon(|\mu|_p(S) + 2)
\end{aligned}
$$

and therefore $p(f_n - f_m) \le 4\sup_{E\in\Sigma} p(\int_E (f_n - f_m) d\mu) \to 0$ as $n,m \to \infty$. By Theorem 10, $f \in I$ and $p(f_n - f) \to 0$ for every $p$, that is, $f_n \to f$ in $I_S$. We have $f_n \to f$ in $L_1$ and in $I_S$; hence $Tf = \lim_n Tf_n = \lim_n \int_S f_n d\mu = \int_S f d\mu$.

CHAPTER III
SET VALUED INTEGRALS

## III.1 The Space of Subsets of X

Let X be a Hausdorff locally convex topological vector space, $\psi$ a base of neighborhoods of zero consisting of balanced convex sets. We begin with the following elementary lemma.

Lemma 1. A set $A \subset X$ is convex iff $(r + s)A = rA + sA$ for all $r, s \geq 0$.

Proof: Suppose A is convex, $r, s \geq 0$. If $r = 0$ or $s = 0$, the result is immediate. It is also clear that $(r+s)A \subset rA+sA$. Assume $r>0$, $s>0$. Since $r/(r+s)>0$ and $s/(r+s)>0$ and $(r/(r+s))+(s/(r+s))=1$ we have

$$(r/(r + s))a_1 + (s/(r+s))a_2 \in A$$

for all $a_1, a_2 \in A$. Then

$$ra_1 + sa_2 = (r + s)[(r/(r + s))a_1 + (s/(r + s))a_2] \in (r + s)A$$

thus $rA+sA \subset (r+s)A$.

Corollary 2. If A is convex, then $nA = \Sigma_{i=1}^{n} A_n$.

Now let $P_0(X)$ be the power set of X less the empty set and for $U \in \psi$ define $E_U = \{(A,B): A \subset B + U, B \subset A + U\}$. Let $\tau$ be the topology on X and $\tau^\wedge$ the topology induced on $P_0(X)$ by the uniformity $\{E_U : U \in \psi\}$.

Lemma 3. If U is balanced and $(0,A)$ is not an element of $E_U$, then A is not contained in U.

Proof: Suppose $A \subset U$. Let $x \in A$. Then $x \in U$ and since U is balanced we have $-x \in U$. Then $0 = x + (-x) \in A + U$; hence $(0,A) \in E_U$. It follows that A is not contained in U.

Corollary 4. Let $(A_n)$ be a sequence in $P_0(X)$. $A_n \to 0$ in $\tau\hat{}$ iff for each $U \in \psi$ there is an integer $n_U$ such that $A_n \subset U$ for all $n \geq n_U$.

Proof: If $A_n \to 0$ in $\tau\hat{}$, then given $U \in \psi$ there is $n_U$ such that $(A_n,0) \in E_U$ for $n \geq n_U$. In particular, $A_n \subset 0 + U = U$. The converse follows from Lemma 3.

Lemma 5. The operations $(A,B) \to A + B$ and $A \to co(A)$ are continuous (uniformly) in $\tau\hat{}$.

Proof: If $(A,A'),(B,B') \in E_U$, then $A' + B' \subset A + U + B + U = A + B + 2U$ and $A + B \subset A' + B' + 2U$; hence $(A + B, A' + B') \in E_{2U}$. Also $A \subset A' + U \subset co(A') + U$. Since U is convex so is the set $co(A') + U$; hence $co(A) \subset co(A') + U$. Similarly $co(A') \subset co(A) + U$ and we have $(co(A),co(A')) \in E_U$.

Lemma 6 ([Dr76]). Let $A,B,C \in P_0(X)$ with B bounded and C convex. If $A + B \subset \overline{C + B}$, then $A \subset \overline{C}$.

Proof: Let $U \in \psi$, $\alpha > 0$ such that $\beta B \subset U$ for $|\beta| \leq \alpha$. Let $a \in A$, $b_0 \in B$. Since $A + B \subset C + B + U$ there are elements $c_1 \in C$, $b_1 \in B$,

$u_1 \in U$ such that $a + b_0 = c_1 + b_1 + u_1$. By induction choose sequences $(c_i)$, $(b_i)$, $(u_i)$ from C, B, and U respectively such that $a + b_{i-1} = c_i + b_i + u_i$. For each n we have

$$na + \Sigma_{i=0}^{n-1} b_i = \Sigma_{i=1}^{n} (c_i + b_i + u_i) \text{ or equivalently}$$

$$a = (\Sigma_{i=1}^{n} c_i)/n + (b_n - b_0)/n + (\Sigma_{i=1}^{n} u_i)/n.$$

For n large we have $1/n < \alpha$; hence

$$(b_n - b_0)/n \in (1/n)B + (-1/n)B \quad 2U.$$

Since C and U are convex we have $(\Sigma_{i=1}^{n} c_i)/n \in C$ and $(\Sigma_{i=1}^{n} u_i)/n \in U$ and thus $a \in C + 3U$. Since $U \in \psi$ is arbitrary it follows that $a \in \overline{C}$.

Corollary 7. The class X of nonempty bounded convex subsets of X is an abelian semi-group in which the cancellation laws hold.

Remark. We make no distinction between a set and its closure so that $A=B$ iff $\overline{A} = \overline{B}$.

Lemma 8. If $(A_n)$ is a sequence of convex sets in X such that $A_n \to A$, then $co(A) \quad \overline{A}$.

Proof: Let $U \in \psi$, $x \in co(A)$, $x = \alpha y + (1 - \alpha)z$ for some $\alpha \in [0,1]$ and $y,z \in A$. Choose n such that $(A_n, A) \in E_U$. In particular $A \quad A_n + U$ and there are $y', z' \in A_n$ such that $y - y' \in U$ and $z - z' \in U$. Since $A_n$ is convex we have $x' = \alpha y' + (1 - \alpha)z' \in A_n$. Then

$$x - x' = \alpha(y - y') + (1 - \alpha)(z - z') \in \alpha U + (1 - \alpha)U = U;$$

hence $x \in A_n + U \subset A + 2U$. Since U and x are arbitrary we have $co(A) \quad A$.

Theorem 9. If $(A_n)$ is a sequence of convex sets in X such that $A_n \to A$, then A is convex ($\overline{A}$ is convex).

Theorem 10. If X is complete and $\psi$ is countably generated, then $P_0(X)$ is complete. In particular if X is a complete metrizable space or Banach space, then $P_0(X)$ is metrizable with the Hausdorff metric and is a complete metric space.

Proof: Suppose $\psi$ is generated by $\{U_1, U_2, U_3, \ldots\}$ $\psi$. Without loss of generality assume $U_1 \supset U_2 \supset U_3 \ldots$. Now let $(A_n)$ be cauchy in $P_0(X)$ and $U \in \psi$. For each k choose $N_k$ such that

$$(A_n, A_m) \in E_{(1/2^{k+1})(U_k \cap U)} \quad \text{for } n,m \geq N_k.$$

Let $n_1 > N_1$, $x_1 \in A_{n_1}$. Choose $n_2 > \max\{N_2, n_1\}$. Then $(A_{n_1}, A_{n_2}) \in E_{(1/4)U_1}$; hence $x_1 - x_2 \in U/4$ for some $x_2 \in A_{n_2}$. Suppose $n_1 < n_2 < \ldots < n_k$, $x_1, x_2, \ldots x_k$ are chosen. Then choose $n_{k+1} > \max\{N_{k+1}, n_k\}$ and $x_{k+1} \in A_{n_{k+1}}$ such that $x_k - x_{k+1} \in (1/2^{k+1})U_k$. Thus we choose sequences $(n_k)$ and $(x_k)$ with $(n_k)$ increasing and $x_k \in A_{n_k}$ for each k. Fix k and let $j > 1 \geq k$. Then

$$x_i - x_j = \Sigma_{l=1}^{j-1}(x_l - x_{l+1}) \in \Sigma_{l=1}^{j-1}(1/2^{l+1})U_l$$

$$\Sigma_{l=i}^{j-1}(1/2^{l+1})U_k = (\Sigma_{l=i}^{j-1} 1/2^{l+1})U_k \subset U_k;$$

hence $(x_k)$ is cauchy in X. Let $x = \lim_k x_k$. Then

$$x \in A = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} A_m} = \overline{\lim}_n A_n.$$

For $z_k = \Sigma_{i=1}^{k-1}(x_i - x_{i+1})$ we have $z_k \in \Sigma_{i=1}^{k-1}(1/2^{i+1})U \subset U$ and

$$x_1 - x = x_1 - \lim_k x_k = \lim_k (x_1 - x_k) = \lim_k z_k \in 2U.$$

Since $x_1$ is arbitrary in $A_{n_1}$ it follows that $A_{n_1} \subset A + 2U$. For $n \geq N_1$ we have

$$(A_n, A_{n_1}) \in E_{(1/4)(U_1 \cap U)}$$

hence $A_n \subset A_{n_1} + U \subset A + 3U$. Now let $x \in A$. Then $x \in \overline{\bigcup_{m=N_1}^{\infty} A_m}$ so there is $m_0 \geq N_1$ and $y_{m_0} \in A_{m_0}$ such that $x - y_{m_0} \in U$. Then for $n \geq N_1$ we have

$$(A_n, A_{m_0}) \in E_{(1/4)(U_1 \cap U)};$$

hence $y_{m_0} - z_n \in U$ for some $z_n \in A_n$. Then

$$x - z_n = x - y_{m_0} + y_{m_0} - z_n \in 2U$$

and thus $A \subset A_n + 2U$. We have $(A, A_n) \in E_{3U}$ for $n \geq N_1$ thus $A_n \to A$ in $\tau^{\wedge}$ and $P_0(X)$ is complete.

Definition 11. Let $(A_n)$ be a sequence of sets and let $S_k = \Sigma_{n=1}^{k} A_n$. If $(S_k)$ is convergent in $\tau^{\wedge}$, then the limit is denoted by $\Sigma_n A_n$ and is called a convergent series.

Theorem 12. If $\Sigma_n A_n$ converges, then so does $\Sigma_n co(A_n)$ and $\Sigma_n co(A_n) = co(\Sigma_n A_n)$.

Proof: The result follows immediately from the additivity and continuity of the map $A \to co(A)$.

Lemma 13. Let $(A_n)$ be a sequence of sets, $S_k = \Sigma_{n=1}^{k} A_n$, and $A = \{\Sigma_{n=1}^{\infty} a_n : a_n \in A_n\}$. If $S_k \to S$ and A is nonempty, then $S = A$ $(\overline{S} = \overline{A})$.

Proof: Let $a \in A$, $U \in \psi$. There are $a_n \in A_n$ such that $a = \Sigma_n a_n$; let $s_k = \Sigma_{n=1}^{k} a_n$. Then $s_k \in S_k$ and $a = \lim_k s_k$. Choose $k_U$ such that $(S_k, S) \in E_U$ for $k \geq k_U$. Then for $k \geq k_U$ we have $s_k \in S_k \subset S + U$; hence $a \in S + U + U = S + 2U$. It follows that $a \in \overline{S}$.

Conversely, let $s \in S$ and $U \in \psi$. Since A is nonempty let $a \in A$, $a = \Sigma a_n$. Choose $k_U$ such that $(S_k, S) \in E_u$ and $\Sigma_{n=k+1}^{\infty} a_n \in U$ for $k \geq k_U$. Then $s \in S \subset S_{k_U} + U$; hence there is an element $s_{k_U} \in S_{k_U}$ such that $s - s_{k_U} \in U$. Let $y = s_{k_U} + \Sigma_{n=k_U+1}^{\infty} a_n$. Then $y \in A$ and

$$s - y = s - s_{k_U} + s_{k_U} - y \in U + U = 2U.$$

It follows that $s \in y + 2U \subset A + 2U$; hence $s \in A$.

Definition 14. Direct the finite subsets $\pi$ of positive integers by inclusion. A sequence $(A_n)$ of sets in X is unconditionally summable if the limit $\lim_{\pi} \Sigma_{n \in \pi} A_n$ exists. The limit is denoted by $\Sigma A_n$.

Lemma 15. If $(A_n)$ is an unconditionally summable sequence of bounded sets, then for each $U \in \psi$ there is a set $\pi_U$ such that $\Sigma_{n \in J} A_n \subset U$ for $J \cap \pi_U = \emptyset$. Hence $\Sigma A_n$ is bounded. (Boundedness is essential. See the counterexample following Theorem 23.)

Proof: Let $U \in \psi$, $A = \lim_{\pi} \Sigma_{n\in\pi} A_n$. Choose $\pi_U$ such that $(A, \Sigma_{m\in\pi} A_n) \in E_U$ for $\pi \geq \pi_U$. In particular $A \subset \Sigma_{n\in\pi_U} A_n + U$. Let $J \cap \pi_U = \emptyset$, $\pi' = J \cup \pi_U$. Then $\pi' \geq \pi_U$ so that $(A, \Sigma_{n\in\pi'} A_n) \in E_U$. We have

$$\Sigma_{n\in\pi_U} A_n + \Sigma_{n\in J} A_n = \Sigma_{n\in\pi'} A_n \subset A + U \subset \Sigma_{n\in\pi_U} A_n + 2U.$$

Since $\Sigma_{n\in\pi_U} A_n$ is bounded and 2U is convex we have $\Sigma_{n\in J} A \subset 2U$ by Lemma 6.

Remark 16. If $P_0(X)$ is complete (X complete and metrizable for example), then the converse of Lemma 15 is true. Lemma 15 and its converse form the cauchy criterion for convergence of series of sets in the present setting.

Theorem 17. A sequence $(A_n)$ is unconditionally summable iff the series $\Sigma_n A_{\sigma(n)}$ converges for every rearrangement $\sigma$.

Proof: Sufficiency of unconditional summability for the convergence of every rearrangement is clear. For the converse suppose each rearrangement is convergent but that $(A_n)$ is not unconditionally summable. Let $A = \Sigma_{n=1}^{\infty} A_n$ in that order. Since $(A_n)$ is not unconditionally summable to A there is a set $U \in \psi$ such that for each $\pi$ there is $\pi' \geq \pi$ such that $(\Sigma_{n\in\pi'} A_n, A)$ is not in $E_U$. Choose $k_0$ such that

(1) $(\Sigma_{n=1}^{k} A_n, A) \in E_{U/2}$ for $k \geq k_0$.

Let $\pi_1 = \{1,2,\ldots,k_0\}$ and choose $\pi_1' > \pi_1$ such that $(\Sigma_{n\in\pi_1'}A_n,A)$ is not in $E_U$. Let $\pi_2 = \{1,2,\ldots,\max \pi_1'\}$ and choose $\pi_2'$ such that $(\Sigma_{n\in\pi_2'}A_n,A)$ is not in $E_U$.

Proceed to choose sets $\pi_3,\pi_3',\pi_4,\pi_4',\ldots$. Taking in order the elements of the sets $\pi_1,\pi_1'\ \pi_1,\pi_2\ \pi_1',\pi_2'\ \pi_2,\ldots$ specifies a rearrangement $\sigma$ among whose partial sums are the sets $\Sigma_{n\in\pi_m}A_n$ and $\Sigma_{n\in\pi_m'}A_n$ and such that

(2) $(\Sigma_{n\in\pi_m'}A_n,A)$ is not in $E_U$ for all m.

If $(\Sigma_{n\in\pi_m}A_n,\Sigma_{n\in\pi_m'}A_n) \in E_{U/2}$, then by (1) we have

$$\Sigma_{n\in\pi_m'}A_n \subset \Sigma_{n\in\pi_m}A_n + U/2 \subset A + U/2 + U/2 = A + U \text{ and}$$

$$A \subset \Sigma_{n\in\pi_m}A_n + U/2 \subset \Sigma_{n\in\pi_m'}A_n + U/2 + U/2 = \Sigma_{n\in\pi_m'}A_n + U$$

contradicting (2). It follows that $(\Sigma_{n\in\pi_m}A_n,\Sigma_{n\in\pi_m'}A_n)$ is not in $E_{U/2}$ for all m thus $(\Sigma_{n=1}^{\infty}A_{\sigma(n)})$ is not cauchy.

Definition 18. A sequence $(A_n)$ in $P_0(X)$ is unconditionally selection summable to A X if $\Sigma_n a_n$ is unconditionally convergent for each sequence (selection) $(a_n)$ with $a_n \in A_n$ and if A is the set of all such sums.

Lemma 19. If $(A_n)$ is unconditionally selection summable, then for each $U \in \psi$, there is a set $\pi_U$ such that $(0,\Sigma_{n\in J}A_n) \in E_U$ for $J \bigcap \pi_U = \emptyset$.

Proof: If not, then there is a set $U \in \psi$ such that for each $\pi$ there is a set $J \subset N\backslash\pi$ such that $(0,\Sigma_{n\in J}A_n)$ is not in $E_U$. Let $J_1 \subset N$ such

that $(0,\Sigma_{n\in J_1} A_n)$ is not $E_U$ and suppose $J_2,J_3,\ldots,J_k$ are chosen. Then choose $J_{k+1}\subset N\setminus\bigcup_{n=1}^{k}J_n$ such that $(0,\Sigma_{n\in J_{k+1}} A_n)$ is not in $E_U$. Thus we choose, by induction, a sequence $(J_k)$ such that $(0,\Sigma_{n\in J_k} A_n)$ is not in $E_U$ for all k. By Lemma 3, $\Sigma_{n\in J_k} A_n$ is not contained in U for each k so that there are elements $a_n \in A_n$, $n \in J_k$, such that $\Sigma_{n\in J_k} a_n$ is not in U. For $n \in N\setminus\bigcup_{k=1}^{\infty}J_k$ let $a_n \in A_n$ be arbitrary. Then $(a_n)$ is a selection of $(A_n)$ whose finite sums do not satisy the cauchy criterion for convergence; hence $(A_n)$ is not unconditionally selection summable, a contradiction.

Theorem 20. If X is complete, then a sequence $(A_n)$ in $P_0(X)$ is unconditionally selection summable if and only if for each $U \in \psi$ there is a set $\pi_U$ such that $(0,\Sigma_{n\in J} A_n) \in E_U$ for $J\cap\pi_U = \emptyset$.

Proof: The forward direction is Lemma 19. For the converse let $(a_n)$ be a selection of $(A_n)$ and let $U \in \psi$. Choose $\pi_U$ such that $(0,\Sigma_{n\in J} A_n) \in E_U$ for $J\cap\pi_U = \emptyset$. Then for $J\subset N\setminus\pi_U$ we have $\Sigma_{n\in J} a_n \in \Sigma_{n\in J} A_n$ U so that the finite sums of $(a_n)$ satisfy the cauchy criterion in X which is complete.

Corollary 21. If X is complete and $(B_n)$ is an unconditionally summable sequence of bounded sets, then $(B_n)$ is unconditionally selection summable.

Proof: The result follows from the theorem and from Lemma 15.

Lemma 22. If $(A_n)$ is unconditionally selection summable to A, then $(A_n)$ is unconditionally summable to A.

Proof: Let $U \in \psi$. By Lemma 19 there is a set $\pi_U$ such that

(1) $(0, \Sigma_{n\in J} A_n) \in E_U$ for $J \cap \pi_U = \emptyset$.

Let $\pi \geq \pi_U$ and let $a_n \in A_n$ for $n \in \pi$. If $a_n \in A_n$ for $n \in N \setminus \pi$ are chosen, then $(a_n)$ is a selection of $(A_n)$ which by hypothesis is unconditionally convergent to an element $a \in A$. Then there is a set $\pi' > \pi$ such that $a - \Sigma_{n\in\pi'} a_n \in U$. By (1) we have

$$a - \Sigma_{n\in\pi} a_n = (a - \Sigma_{n\in\pi'} a_n) + (\Sigma_{n\in\pi'} a_n - \Sigma_{n\in\pi} a_n) \in U + U = 2U.$$

Since U is balanced we also have $\Sigma_{n\in\pi} a_n - a \in 2U$; hence $\Sigma_{n\in\pi} a_n \in a + 2U \subset A + 2U$. Since $(a_n)_{n\in\pi}$ is arbitrary in $(A_n)_{n\in\pi}$ we have

(2) $\Sigma_{n\in\pi} A_n \subset A + 2U$.

Now let $a \in A$. There is a selection $(a_n)$ such that $a = \Sigma a_n$. There is a set $\pi' > \pi$ such that $a - \Sigma_{n\in\pi'} a_n \in U$. By (1) we have $a - \Sigma_{n\in\pi} a_n \in 2U$; hence $a \in \Sigma_{n\in\pi} A_n + 2U$ which together with (2) shows that $(A, \Sigma_{n\in\pi} A_n) \in E_{2U}$.

Theorem 23. If X is complete, then a sequence $(B_n)$ of bounded sets in $P_0X)$ is unconditionally selection summable to a set B if and only if it is unconditionally summable to B.

Proof: The forward direction is Lemma 22. For the converse suppose $(B_n)$ is unconditionally summable to B. By Corollary 21 $(B_n)$ is unconditionally selection summable to a set B', thus $(B_n)$ is

unconditionally summable to B'. It follows that B and B' have the same closure.

Remark 24 ([So85]). Boundedness of the sets $B_n$ is essential. Let $A_n$ be the linear subspace $\{(x,y): y = x/n\}$ of $R^2$. For n different from m we have $A_n + A_m = R^2$ and hence $(A_n)$ is unconditionally summable to $R^2$. However, the sequence $(a_n)$ where $a_n=(n,1)$ while satisfying $a_n \in A_n$ for all n is clearly not unconditionally summable.

Definition 25. Let p be a continuous seminorm on X. $U_p$ denotes the set $\{x: p(x) < 1\}$. If $U \in \psi$, then $p_U$ denotes the gauge of U. We have $U_{p_U} = U$ and $p_{U_p} = p$. Let $A,B \subset X$. $p(A)$ denotes the supremum $\sup_{a\in A} p(a)$. It is clear that $p(A + B) \leq p(A) + p(B)$ and that $p(\alpha A) = |\alpha| p(A)$ for $\alpha$ scalar. Also, $A_\beta \to 0$ if and only if $p(A_\beta) \to 0$ for every continuous seminorm p. $D_p(A,B)$ denotes the infimum $\inf\{r > 0: A \subset B + rU_p, B \subset A + rU_p\}$ where we agree that $\inf\emptyset = \infty$. If A and B are bounded and p is a continuous seminorm, then $U_p \in \psi$ and A - B is bounded; hence there is an $r < \infty$ such that $A - B \subset rU_p$ and $B - A \subset rU_p$. Then $A \subset B + A - B \subset B + rU_p$ and $B \subset A + rU_p$ so that $D_p(A,B) \leq r < \infty$. For $U \in \psi$, $p = p_U$, and $\alpha > 0$, one has $(A,B) \in E_{\alpha U}$ if and only if $D_p(A,B) \leq \alpha$.

Lemma 26. A set $B \subset X$ is bounded if and only if $p(B) < \infty$ for every continuous seminorm p on X.

Proof: Suppose B is bounded, p continuous. There is a number $\alpha > 0$ such that $\beta B \subset U_p$ for $|\beta| < \alpha$. Then $\alpha p(B) = p(\alpha B) \leq p(U_p) \leq 1$; hence $p(B) \leq 1/\alpha < \infty$. Conversely, let $U \in \psi$, $p = p_{U/2}$. Then

$p(B) = M < \infty$ and for $|\beta| \leq 1/M$ we have $p(\beta B) = |\beta| p(B) \leq 1$; hence $\beta B \subset \overline{U}/2 \subset U$.

Corollary to Theorem 17. If X is complete and $\Sigma p(A_n) < \infty$ for every continuous p, then $(A_n)$ is an unconditionally summable sequence of bounded sets and therefore also unconditionally selection summable.

Lemma 27. If $(A_n)$ is unconditionally summable or unconditionally selection summable, then $p(\Sigma A_n) \leq \Sigma p(A_n)$.

Proof: Without loss of generality suppose $(A_n)$ is unconditionally summable. Let $\alpha > 0$, $U_\alpha = \{x : p(x) < \alpha\}$. Then $U_\alpha \in \psi$ and there is a set $\pi_\alpha$ such that $(\Sigma_{n\in\pi} A_n, \Sigma A_n) \in E_{U_\alpha}$ for $\pi \geq \pi_\alpha$. Hence $\Sigma A_n \subset U_\alpha + \Sigma_{n\in\pi_\alpha} A_n$ and we have

$$p(\Sigma A_n) \leq p(U_\alpha) + p(\Sigma_{n\in\pi_\alpha} A_n) \leq \alpha + \Sigma_{n\in\pi_\alpha} p(A_n) \leq \alpha + \Sigma p(A_n).$$

Since $\alpha > 0$ is arbitrary it follows that $p(\Sigma A_n) \leq \Sigma p(A_n)$.

Lemma 28. $A_n \to A$ in $P_0(X)$ if and only if $D_p(A_n, A) \to 0$ for every continuous seminorm p.

Proof: Suppose $A_n \to A$ and let p be a continuous seminorm on X. For $\alpha > 0$ and $U = U_p$ there is an integer $n_U$ such that $(A_n, A) \in E_{\alpha U}$ for $n \geq n_U$ or $D_p(A_n, A) \leq \alpha$. It follows that $\lim_{n\to\infty} D_p(A_n, A) = 0$. Conversely, suppose $D_p(A_n, A) \to 0$ for every p. For $U \in \psi$, $p = p_{U/3}$,

and $\alpha > 0$, choose $n_\alpha$ such that $D_p(A_n,A) < \alpha$ for $n \geq n_\alpha$. Then $(A_n,A) \in E_{\alpha U/\alpha} = E_U$ for $n \geq n_\alpha$. It follows that $A_n \to A$ in $P(X)$.

Lemma 29. $D_p(\alpha A,\alpha B) = |\alpha| D_p(A,B)$.

Proof:
$$\begin{aligned} D_p(\alpha A,\alpha B) &= \inf\{r > 0 : \alpha A \subset \alpha B + rU_p, \alpha B \subset \alpha A + rU_p\} \\ &= \inf\{r > 0 : A \subset B + (r/|\alpha|)U_p, B \subset A + (r/|\alpha|)U_p\} \\ &= |\alpha| \inf\{r/|a| > 0 : A \subset B + (r/|\alpha|)U_p, B \subset A + (r/|a|)U_p\} \\ &= |\alpha| D_p(A,B). \end{aligned}$$

Lemma 29a. Let $u$ be a bounded linear operator on $X$ into a locally convex space $Y$ with seminorms $q$. For $A,B$ bounded we have $D_q(uA,uB) \leq |u|_{pq} D_p(A,B)$ where $|u|_{pq} = \sup_{p(x)\leq 1} q(ux)$.

Proof: Let $r > D_p(A,B)$. Then $A \subset B + rU_p$ and for every $a \in A$ there is an element $b \in B$ such that

$$q(ua - ub) = q(u(a - b)) \leq |u|_{pq} p(a - b) < |u|_{pq} r;$$

hence $uA \subset uB + |u|_{pq} rU_q$. Similarly $uB \subset uA + |u|_{pq} rU_q$; hence $D_q(uA,uB) \leq |u|_{pq} r$. Since $r > D_p(A,B)$ is arbitrary, the conclusion follows.

Lemma 30. $D_p(A + B,C + D) \leq D_p(A,C) + D_p(B,D)$.

Proof: Let $r > D_p(A,C) + D_p(B,D)$, and let $s > D_p(A,C)$ and $t > D_p(B,D)$ such that $r = s + t$. Let $a + b \in A + B$. Since $D_p(A,C) < s$ we have $a \in A \subset C + sU_p$ so that there is an element $c \in C$ such that $a - c \in sU_p$. Then $p(a - c) < s$. Similarly there is an element $d \in D$ such that $p(b - d) < t$. Then

$p(a + b - [c + d]) = p(a - c + b - d) \leq p(a - c) + p(b - d) < s + t = r$

and it follows that $a + b \in c + d = rU_p$. Since $a + b$ is arbitrary in $A + B$ it follows that $A + B \subset C + D + rU_p$. Similarly we have $C + D \subset A + B + rU_p$; hence $D_p(A + B, C + D) \leq r$.

Lemma 31. If $D_p(A_n, A) \to 0$, then for each $a \in A$ there is a sequence $(a_n)$ with $a_n \in A_n$ such that $p(a_n - a) \to 0$.

Proof: Let $a \in A$ and for each $n$ let $r_n = D_p(A_n, A) + 1/n$. Since $r_n > D_p(A_n, A)$ there is an element $a_n \in A_n$ such that $p(a_n - a) \leq r_n \to 0$.

Lemma 32. If $D_p(A,B) = 0$ for every $p$, then $A$ and $B$ have the same closure.

Proof: For each $U \in \psi$ we have $D_{p_U}(A,B) < 1$ thus $A \subset B + U$ and $B \subset A + U$.

Definition 33. If $X$ is a normed space, then for $p(x) = |x|$ we denote $D_p(A,B)$ by $D(A,B)$. $D$ is the well known Hausdorff distance. By definition $D(A,B) = \inf\{r > 0 : A \subset B + rS_1, B \subset A + rS_1\}$ where $S_1 = \{x : |x| < 1\}$. By definition 25 we have $(A,B) \in E_{\alpha S_1}$ if and only if $D(A,B) \leq \alpha$. Since the spheres $\alpha S_1$ generate the topology $\tau$ of $X$ it follows that the topology induced on $P_0(X)$ by $D$ is $\tau\hat{}$. By taking $\alpha = 1/n$ for $n = 1,2,\ldots$ we see from Theorem 10 that $P_0(X)$ is complete if $X$ is complete. For $A,B \in P_0(X)$ the Hausdorff semidistance between $A$ and $B$ is the number $\delta(A,B) = \inf\{r > 0 : A \quad B + rS_1\}$.

Lemma 34. $D(A,B) = \max\{\delta(A,B),\delta(B,A)\}$ and Lemmas 29 and 30 hold for $\delta$. Also, $\delta(A,B) = 0$ if and only if $A \subset \overline{B}$.

Corollary 35. $D(A,B) = 0$ if and only if A and B have the same closure.

Definition 36. $\tilde{X}$ will denote the set of nonempty bounded convex subsets of X, and for $x^* \in X^*$, $A \subset X$, $x^*(A)$ is the set of images $\{x^*x : x \in A\}$.

Lemma 37. For $A,B \in P_0(X)$ with convex closures, $x^*(A) = x^*(B)$ for every $x^* \in X^*$ implies A and B have the same closure.

Proof: Suppose A and B do not share the same closure. Then there is without loss of generality an element $a \in A - B$. Since $\{a\}$ and B are disjoint closed sets with one of them compact, there is a functional $x^* \in X^*$ separating them strictly, that is $x^*a$ is not contained in the closure of $x^*(B)$. Since $x^*(A) = x^*(B)$ we have $x^*a \in x^*(B)$, a contradiction. The conclusion follows.

## III.2 Set Valued Set Functions

Let S be a set with an algebra $\Sigma$ of subsets. A function $J:\Sigma \to P_0(X)$ is additive if $J(\emptyset) = 0$ and $J(E \cup F) = J(E) + J(F)$ for E,F disjoint in $\Sigma$. Given a seminorm p on X, the p-variation of J on a set $E \in \Sigma$ is given by $v_p(J,E) = \sup\Sigma_i p(J(E_i))$ where the supremum is taken over all finite pairwise disjoint sequences $(E_i)$ in $\Sigma$ with $E_i \subset E$.

Theorem 1. The p-variation of an additive function is additive.

Proof: Let E,F be disjoint elements of $\Sigma$. Let $(A_i)$ be a finite disjoint sequence in $\Sigma$ such that $A_i \subset E \cup F$ and denote by $E_i, F_i$ the sets $A_i \cap E$ and $A_i \cap F$, respectively. Then

$$\Sigma p(J(A_i)) = \Sigma p(J(E_i) + J(F_i)) \leq \Sigma p(J(E_i)) + \Sigma p(J(F_i))$$

$$\leq v_p(J,E) + v_p(J,F)$$

thus $v_p(J,E \cup F) \leq v_p(J,E) + v_p(J,F)$. If $v_p(J,E \cup F) = \infty$, then equality holds; otherwise, let $\varepsilon > 0$. Since $v_p(J,E) \leq v_p(J,E \cup F) < \infty$ and $v_p(J,F) \leq v_p(J,E \cup F) < \infty$ we may choose finite sequences $(E_i)$ and $(F_j)$ from $\Sigma$ such that $E_i \subset E$, $F_j \subset F$, and $\Sigma_i p(J(E_i)) > v_p(J,E) - \varepsilon$ and $\Sigma_j p(J(F_j)) > v_p(J,F) - \varepsilon$. Then

$$v_p(J,E) + v_p(J,F) < 2\varepsilon + \Sigma_i p(J(E_i)) + \Sigma_j p(J(F_j)) \leq 2\varepsilon + v_p(J,E \cup F).$$

Since $\varepsilon > 0$ is arbitrary it follows that $v_p(J,E) + v_p(J,F) \leq v_p(J,E \cup F)$.

Definition 3. An additive function $J:\Sigma \to P_0(X)$ is strongly additive if $J(E_n) \to 0$ for every disjoint sequence $(E_n)$ in $\Sigma$. A strongly additive function is bounded (see Rickart [Ri44]).

Lemma 4. J is strongly additive if and only if for each disjoint sequence $(E_n)$ and $U \in \psi$ there is a finite set $\pi_U$ of positive integers such that $\Sigma_{n \in I} J(E_n) \subset U$ for all finite sets I disjoint from $\pi_U$.

Proof: Suppose J is strongly additive but that for some disjoint sequence $(E_n)$ there is an element $U \in \psi$ such that for every $\pi$ there is a set I disjoint from $\pi$ such that $\Sigma_{n \in I} J(E_n)$ is not contained in U. Let $I_1$ be such a set for $\pi = \emptyset$ and if $I_2, \ldots, I_k$ are chosen, then

choose $I_{k+1}$ disjoint from $\bigcup_{i=1}^{k} I_i$ such that $\Sigma_{n\in I_{k+1}} J(E_n)$ is not contained in U. Then for $F_k = \bigcup_{n\in I_k} E_n$, $(F_k)$ is a disjoint sequence in $\Sigma$ such that $J(F_k) \not\to 0$. Conversely, given U, let $\pi_U$ be a finite set such that $\Sigma_{n\in I} J(E_n) \subset U$ for I disjoint from $\pi_U$. For all n not in $\pi_U$ we have $J(E_n) = \Sigma_{k\in\{n\}} J(E_k) \subset U$. It follows that $J(E_n) \to 0$.

Corollary 5. If $(J(E_n))$ is an unconditionally selection summable sequence for every disjoint sequence $(E_n)$, then J is strongly additive. (See Lemma 1.19.)

Corollary 5a. If J is bounded-valued and $(J(E_n))$ is unconditionally summable for every disjoint $(E_n)$, then J is strongly additive. (See Lemma 1.15.)

Corollary 6. If X is complete, then J is strongly additive if and only if $(J(E_n))$ is unconditionally selection summable for every disjoint sequence $(E_n)$. (See Lemma 1.20.)

Corollary 6a. If X is complete and J is bounded-valued, then J is strongly additive if and only if $(J(E_n))$ is unconditionally summable for every disjoint $(E_n)$.

Corollary 6b. If X is complete and $\Sigma p(J(E_n)) < \infty$ for every disjoint sequence $(E_n)$ and every p, then J is strongly additive.

Corollary 6c. If X is complete and J has finite p-variation for every p, then J is strongly additive.

Definition 7. Let $J:\Sigma \to P_0(X)$, $\mu:\Sigma \to [0,\infty]$ be additive. $J$ is absolutely continuous with respect to $\mu$ or simply, $\mu$-continuous, if $\lim_{\mu(E)\to 0} J(E) = 0$.

Theorem 8 (Drewnowski [Dr76]). Let $(J_n)$ be a sequence of strongly additive $\mu$-continuous functions on $\Sigma$ with values in the bounded nonempty subsets of $X$. If $\lim_{n\to\infty} J_n(E)$ exists for every $E \in \Sigma$, then $\lim_{\mu(E)\to 0} J_n(E) = 0$ uniformly for $n = 1,2,\ldots$.

Definition 9. An additive function $J:\Sigma \to P_0(X)$ is countably additive if for every disjoint sequence $(E_n)$ in $\Sigma$ with union $E \in \Sigma$, the sequence $(J(E_n))$ is unconditionally summable to $J(E)$.

Theorem 10. Let $J:\Sigma \to P_0(X)$ be additive. Then $J$ is countably additive if and only if $J(E_n) \to J(E)$ in $P_0(X)$ for every increasing sequence $(E_n)$ in $\Sigma$ with union $E \in \Sigma$.

Proof: Suppose $J$ is countably additive and $(E_n)$ increases to $E$ in $\Sigma$. For $n$ different from 0 let $A_n = E_n - E_{n-1}$, $E_0 = \emptyset$. Then $(A_n)$ is a disjoint sequence in $\Sigma$ and $A_n = E$; hence $(J(A_n))$ unconditionally summable to $J(E)$. By Theorem 1.17 we have in particular $J(E) = \Sigma_{n=1}^{\infty} J(A_n)$. Thus

$$J(E_n) = J(\bigcup_{i=1}^{n} A_i) = \Sigma_{i=1}^{n} J(A_i) \to J(E).$$

Conversely, let $(A_n)$ be disjoint in $\Sigma$ with union $A \in \Sigma$. Let $E_m = \bigcup_{n=1}^{m} A_n$ for all $m$. Then $(E_m)$ increases to $A$ so that

$$\Sigma_{n=1}^{m} J(A_n) = J(\bigcup_{n=1}^{m} A_n) = J(E_m) \to J(A).$$

If $\sigma(n)$ is any rearrangement, then the same argument shows that $\Sigma_{n=1}^{m} J(A_{\sigma(n)}) \to J(A)$. It now follows from Theorem 1.17 that $(J(A_n))$ is unconditionally summable to $J(A)$.

Theorem 11. A bounded valued countably additive function $J:\Sigma \to P_0(X)$ on a $\sigma$-algebra is strongly additive.

Proof: Let $(E_n)$ be a disjoint sequence in $\Sigma$. Then $E_n \in \Sigma$ and $(J(E_n))$ is unconditionally summable to $J(\bigcup E_n)$. Let $U \in \psi$. By Lemma 1.16 there is a set $\pi_U$ such that $\Sigma_{n\in I} J(E_n) \subset U$ for $I \cap \pi_U = \emptyset$. In particular for all $n \in N\backslash\pi_U$ we have $J(E_n) \subset U$. It follows that $J(E_n) \to 0$.

Theorem 12. If $J:\Sigma \to P_0(X)$ is countably additive, then $v_p(J)$ is countably additive for each $p$.

Proof: Let $(E_n)$ be a disjoint sequence $\Sigma$ with union $E \in \Sigma$. For each $m$, since $v_p(J)$ is additive, we have

$$v_p(J,E) \geq v_p(J, \bigcup_{n=1}^{m} E_n) = \Sigma_{n=1}^{m} v_p(J,E_n);$$

hence $v_p(J,E) \geq \Sigma v_p(J,E_n)$. If $(F_i)$ is any finite disjoint sequence in $\Sigma$ with $F_i \subset E$, then by Lemma 1.27

$$\Sigma_i p(J(F_i)) = \Sigma_i p(J(\bigcup_n F_i E_n)) = \Sigma_i p(\Sigma_n J(F_i E_n))$$

$$\leq \Sigma_i \Sigma_n p(J(F_i E_n)) = \Sigma_n \Sigma_i p(J(F_i E_n)) \leq \Sigma_n v_p(J,E_n);$$

hence $v_p(J,E) \leq \Sigma_n v_p(J,E_n)$.

Theorem 13. If $J:\Sigma \to P_0(X)$ is bounded valued and weakly strongly additive, then it is strongly additive.

Proof: By Corollary 5, it is enough to show that $(J(E_n))$ is unconditionally selection summable for every disjoint sequence $(E_n)$ in $\Sigma$. Let $(E_n)$ be a disjoint sequence in $\Sigma$, and let $(x_n)$ be an arbitrary selection of $(J(E_n))$. Since J is weakly strongly additive, for each $x^* \in X^*$, the function $x^*J$ is strongly additive. Furthermore, $x^*J$ is bounded valued and takes its values in $P_0(R)$. Since R is complete it follows from Corollary 6 that $(x^*J(E_n))$ is unconditionally selection summable. Since $(x^*x_n)$ is a selection of $(x^*J(E_n))$, the series $\Sigma_n x^*x_n$ is unconditionally convergent. It follows that any subseries is also unconditionally convergent; hence by the Orlicz-Pettis Theorem (see Roberts [Ro69]) $\Sigma_n x_n$ is unconditionally convergent. Thus $(J(E_n))$ is unconditionally selection summable.

Theorem 14. If $J:\Sigma \to \tilde{X}$ is weakly countably additive, then J is countably additive.

Proof: Let $(E_n)$ be a disjoint sequence in $\Sigma$ with union $E \in \Sigma$. We will show $(J(E_n))$ is unconditionally selection summable to J(E) and hence unconditionally summable to J(E). Let $(x_n)$ be an arbitrary selection of $(J(E_n))$. If $x^* \in X^*$, then $x^*J$ is countably additive and $(x^*J(E_n))$ is unconditionally summable to $x^*J(E)$. Since R is complete it follows from Theorem 1.23 that $(x^*J(E_n))$ is unconditionally selection summable to $x^*J(E)$. Since $(x^*x_n)$ is a selection of $(x^*J(E_n))$ the series $\Sigma_n x^*x_n$ it is unconditionally convergent and

hence every subseries is unconditionally convergent. By the Orlicz-Pettis theorem if follows that $\Sigma_n x_n$ is unconditionally convergent. It remains to show that $J(E) = \{\Sigma_n x_n : x_n \in J(E_n)\}$. Let this latter set be denoted by $\Phi$. Then by definition, $(J(E_n))$ is unconditionally selection summable to $\Phi$ and by Lemma 1.22, Theorem 1.17, and Theorem 1.12 we have

$$\Phi = \Sigma J(E_n) = \Sigma_{n=1}^{\infty} J(E_n) = \Sigma_{n=1}^{\infty} co(J(E_n)) = co(\Sigma_{n=1}^{\infty} J(E_n));$$

thus $\Phi$ is convex. For every $x^* \in X^*$ we have

$$x^* J(E) = \{\Sigma_n \alpha_n : \alpha_n \in x^* J(E_n)\} = \{\Sigma_n x^* x_n : x_n \in J(E_n)\} = x^* \Phi$$

and since both $\Phi$ and $J(E)$ are convex, it follows from Lemma 1.37 that $\Phi = J(E)$.

Theorem 15. Let $J:\Sigma \to \bar{R}$ be additive. If $\sup_{F \subset E} |J(F)| < \infty$, then $v(J,E) < \infty$ and $v(J,E) \leq 4\sup_{F \subset E} |J(F)|$.

Proof: First we note that elements of $\bar{R}$ are finite intervals and in fact may be considered closed. Let $J_i(E) = \inf J(E)$, $J_s(E) = \sup J(E)$. Let E,F be disjoint in $\Sigma$. Since J(E) and J(F) are convex, so is there sum J(E) + J(F); thus it is an interval and since $\inf(J(E) + J(F)) = J_i(E) + J_i(F)$ and $\sup(J(E) + J(F)) = J_s(E) + J_s(F)$ it follows that J(E) + J(F) is the interval $[J_i(E) + J_i(F), J_s(E) + J_s(F)]$. By the additivity of J we have $J(E \cup F) = [J_i(E \cup F), J_s(E \cup F)]$.
It follows that $J_i(E \cup F) = J_i(E) + J_i(F)$ and $J_s(E \cup F) = J_s(E) + J_s(F)$,

so that $J_i$ and $J_s$ are additive real valued functions on $\Sigma$. Now, if $\sup_{F \subset E}|J(F)| < \infty$ it follows that the same is true of $J_i$ and $J_s$ and from elementary measure theory we have $v(J_i,E) \leq 2\sup_{F \subset E}|J_i(F)|$ and $v(J_s,E) \leq 2\sup_{F \subset E}|J_s(F)|$. If $(E_i)$ is a finite disjoint sequence in $\Sigma$ with $E_i \subset E$, then

$$\begin{aligned}\Sigma|J(E_i)| &= \Sigma_{max}\{|J_i(E_i)|, |J_s(E_i)|\} \\ &\leq \Sigma(|J_i(E_i)| + |J_s(E_i)|) \leq v(J_i,E) + v(J_s,E) \\ &\leq 2\sup_{F \subset E}|J_i(F)| + 2\sup_{F \subset E}|J_s(F)| \\ &\leq 4\sup_{F \subset E}|J(F)|.\end{aligned}$$

Definition 16. For a bounded set $A \subset R^2$ let $B(A)$ be the open rectangle (inf ReA, sup ReA) x (inf ImA, sup ImA). It is easy to verify that $B:R^2 \to R^2$ is additive.

Lemma 17. If A is bounded, then $B(A) \subset A + S_{d(A)}(A)$, $(S_{d(A)}(A))$ is the sphere of radius diam(A) about A, also given by $A + d(A)S$ and $|B(A)| \leq 3|A|$.

Proof: Let $(x,y) \in B(A)$. Choose r,s such that $(x,r) \in A$ and $(s,y) \in A$. We have $(x,r) \in A$ and $|(x,y) - (x,r)| \leq |(s,y) - (x,r)| \leq d(A)$ thus $(x,y) \in S_{d(A)}(A)$. Also, $|(x,y)| \leq |(x,y) - (x,r)| + |(x,r)|$ $\leq d(A) + |A| \leq 3|A|$.

Theorem 18. Let $J:\Sigma \to P_0(C)$ be a bounded valued additive function. If $\sup_{F \subset E}|J(F)| \leq \infty$, then $v(J,E) < \infty$ and $v(J,E) \leq 24\sup_{F \subset E}|J(F)|$.

Proof: Define $\hat{J}$ on $\Sigma$ by $\hat{J}(E) = B(J(E))$. Since B and J are additive so is $\hat{J}$. In particular $\mathrm{Re}\hat{J}$ and $\mathrm{Im}\hat{J}$ are additive $\tilde{R}$-valued functions and $\hat{J}(E) = \mathrm{Re}\hat{J}(E) + \mathrm{Im}\hat{J}(E)$. Assume $\sup_{F\subset E}|\hat{J}(F)| < \infty$, By Lemma 17, $\sup_{J\subset E}|\hat{J}(E)| < \infty$ and thus $\sup_{F\subset E}|\mathrm{Re}\hat{J}(F)| < \infty$ and $\sup_{F\subset E}|\mathrm{Im}\hat{J}(F)| < \infty$. Then by Lemma 15 and Lemma 17, we have

$$v(\mathrm{Re}\hat{J},E) \le 4\sup_{F\subset E}|\mathrm{Re}\hat{J}(F)| \le 4\sup_{F\subset E}|\hat{J}(F)| \le 12\sup_{F\subset E}|J(F)|.$$

Similarly, $v(\mathrm{Im}\hat{J},E) \le 12\sup_{F\subset E}|J(F)|$; hence $v(J,E) \le v(\hat{J},E) \le v(\mathrm{Re}\hat{J},E) + v(\mathrm{Im}\hat{J},E) \le 24\sup_{F\subset E}|J(F)|$.

## III.3 Integration

Let S be a set with an algebra $\Sigma$ of subsets on which a nonnegative additive function $\mu$ is defined.

Definition 1. A function $f:S \to P_0(X)$ is simple if it is constant on each of a finite number of disjoint sets of $\Sigma$ with union S. A simple $\tilde{X}$-valued function $f = \Sigma_i 1_{E_i} X_i$ is integrable over $E \in \Sigma$ with respect to $\mu$ if $\mu(EE_i) < \infty$ when $X_i$ is different from $\{0\}$. If f is integrable for every $E \in \Sigma$, we say f is integrable. The integral of f over E is given by $\int_E f d\mu = \Sigma_i \mu(EE_i)X_i$. Since $p(\int_E f d\mu) \le \Sigma_i \mu(EE_i)p(X_i) \le \{\max_i p(X_i)\}\mu(E)$ we see that $\lim_{\mu(E)\to 0} p(\int_E f d\mu) = 0$. The class of simple integrable $\tilde{X}$-valued functions is denoted by $\Gamma$. If f(s) is a real number for all s, then the integral is just the Lebesgue integral. If f(s) is a single vector of a Banach space for all s, then the integral is the Bochner integral.

Lemma 2. For $E \in \Sigma$ fixed, the map $f \to \int_E f d\mu$ is additive and homogenous on $\Gamma$.

Proof: Let $f,g \in \Gamma$, $\alpha$ scalar. Then $f + g = \Sigma_{ij} 1_{E_iF_j}(X_i + Y_j)$ where $f = \Sigma_i 1_{E_i} X_i$ and $g = \Sigma_j 1_{F_j} Y_j$ and $\bigcup E_i = \bigcup F_j = S$. If $X_i + Y_j$ is nonzero, then one of the sets $X_i$ or $Y_j$ is nonzero thus $\mu(E_i) < \infty$ or $\mu(F_j) < \infty$; hence $\mu(E_iF_j) < \infty$. It follows that $f + g \in \Gamma$. Clearly $\alpha f \in \Gamma$. Since $X_i$ and $Y_j$ are convex sets we have by Lemma 1,

$$\begin{aligned}
\int_E (f + g)d\mu &= \Sigma_{ij}\mu(EE_iF_j)(X_i + Y_j) \\
&= \Sigma_{ij}\mu(EE_iF_j)X_i + \Sigma_{ij}\mu(EE_iF_j)Y_j \\
&= \Sigma_i\Sigma_j\mu(EE_iF_j)X_i + \Sigma_j\Sigma_i\mu(EE_iF_j)Y_j \\
&= \Sigma_i(\Sigma_j\mu(EE_iF_j))X_i + \Sigma_j(\Sigma_i\mu(EE_iF_j))Y_j \\
&= \Sigma_i\mu(EE_i)X_i + \Sigma_j\mu(EF_j)Y_j \\
&= \int_E f d\mu + \int_E g d\mu.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_E \alpha f d\mu &= \int_E \alpha(\Sigma_i 1_{E_i} X_i)d\mu = \int_E (\Sigma_i 1_{E_i} \alpha X_i)d\mu \\
&= \Sigma_i\mu(EE_i)\alpha X_i = \alpha\Sigma_i\mu(EE_i)X_i = \alpha\int_E f d\mu.
\end{aligned}$$

Lemma 3. For $f \in \Gamma$, the function $\lambda(E) = \int_E f d\mu$ is additive on $\Sigma$ and $v_p(\lambda,E) = \int_E p(f)d\mu$.

Proof: Let $E,F \in \Sigma$ with $EF = \emptyset$. Then for $f = \Sigma_i 1_{E_i} X_i$ with $X_i$ convex,

$$\begin{aligned}
\int_{E\bigcup F} f d\mu &= \Sigma_i\mu(EE_i \bigcup FE_i)X_i = \Sigma_i(\mu(EE_i) + \mu(FE_i))X_i \\
&= \Sigma_i(\mu(EE_i)X_i + \mu(FE_i)X_i) = \Sigma_i\mu(EE_i)X_i + \Sigma_i\mu(FE_i)X_i \\
&= \int_E f d\mu + \int_F f d\mu.
\end{aligned}$$

Now let $(A_m)$ be a finite disjoint sequence in $\Sigma$ such that $A_m \subset E \in \Sigma$. Then

$$\Sigma_m p(\lambda(A_m)) = \Sigma_m p(\Sigma_i \mu(A_m E_i)X_i) \leq \Sigma_m \Sigma_i \mu(A_m E_i)p(X_i)$$

$$= \Sigma_i \Sigma_m \mu(A_m E_i)p(X_i) = \Sigma_i \mu(\bigcup_m A_m E_i)p(X_i)$$

$$\leq \Sigma_i \mu(EE_i)p(X_i) = \int_E p(f)d\mu.$$

Thus $v_p(\lambda,E) \leq \int_E p(f)d\mu$. Since $(EE_i)$ is a disjoint sequence in $\Sigma$ with $EE_i \subset E$, we have $\Sigma_i p(\lambda(EE_i)) \leq v_p(\lambda,E)$ but

$$\Sigma_i p(\lambda(EE_i)) = \Sigma_i p(\mu(EE_i)X_i) = \Sigma_i \mu(EE_i)p(X_i) = \int_E p(f)d\mu$$

thus $\int_E p(f)d\mu \leq v_p(\lambda,E)$. It now follows that $v_p(\lambda,E) = \int_E p(f)d\mu$.

Lemma 4. Let $f,g \in \Gamma$. Then $D_p(f,g)$ is an integrable scalar function and $D_p(\int_E f d\mu, \int_E g d\mu) \leq \int_E D_p(f,g)d\mu$.

Proof: Let $f = \Sigma_i 1_{E_i} X_i$, $g = \Sigma_j 1_{F_j} Y_j$ where $(E_i)$, $(F_j)$ are finite disjoint sequences in $\Sigma$ both with union $S$. Then it is easy to see that $D_p(f,g) = \Sigma_{ij} 1_{E_i F_j} D_p(X_i,Y_j)$ and thus $D_p(f,g)$ is integrable. Since $X_i$ and $Y_j$ are convex we have, by Lemma 1.29 and Lemma 1.30,

$$D_p(\int_E f d\mu, \int_E g d\mu) = D_p(\Sigma_i \mu(EE_i)X_i, \Sigma_j \mu(EF_j)Y_j)$$

$$= D_p(\Sigma_i \Sigma_j \mu(EE_iF_j)X_i, \Sigma_i \Sigma_j \mu(EE_iF_j)Y_j)$$

$$\leq \Sigma_{ij} D_p(\mu(EE_iF_j)X_i, \mu(EE_iF_j)Y_j)$$

$$= \Sigma_{ij} \mu(EE_iF_j) D_p(X_i, Y_j)$$

$$= \int_E D_p(f,g) d\mu.$$

Lemma 5. Let $(f_n^1), (f_n^2)$ be sequences in $\Gamma$ such that $D_p(f_n^1, f_n^2) \to \mu$ and $\lim_{m,n\to\infty} \int_S D_p(f_n^i, f_m^i) d\mu = 0$ for $i = 1,2$. Then $\lim_{n\to\infty} \int_S D_p(f_n^1, f_n^2) d\mu = 0$.

Proof Let $\phi_n(s) = D_p(f_n^1(s), f_n^2(s))$. Then

$$|\phi_n(s) - \phi_m(s)| \leq D_p(f_n^1(s), f_m^1(s)) + D_p(f_n^2(s), f_m^2(s))$$

and therefore $\lim_{m,n\to\infty} \int_S |\phi_n - \phi_m| d\mu = 0$. It follows that the limit $\phi(E) = \lim_{n\to\infty} \int_E \phi_n d\mu$ exists uniformly for $E \in \Sigma$. Let $(E_k)$ be a sequence in $\Sigma$ such that $\mu(E_k) \to 0$. Since $\lim_{\mu(E)\to 0} \int_E \phi_n d\mu = 0$ we have $\lim_{k\to\infty} \int_{E_k} \phi_n d\mu = 0$. By the Moore-Smith theorem we have

$$\lim_k \phi(E_k) = \lim_k \lim_n \int_{E_k} \phi_n d\mu = \lim_n \lim_k \int_{E_k} \phi_n d\mu = 0.$$

It follows from the introductory remarks concerning the metric space $\Sigma(\mu)$ in section III.7 of Dunford and Schwartz [Du58] that $\lim_{\mu(E)\to 0} \phi(E) = 0$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\phi(E) < \varepsilon$ for $\mu(E) < \delta$. Choose $n_\varepsilon$ such that $|\phi(E) - \int_E \phi_n d\mu| < \varepsilon$ for $n \geq n_\varepsilon$ and

all $E \in \Sigma$. Since $\phi_{n_\varepsilon}$ is simple the set $A = \{s:\phi_{n_\varepsilon}(s) > 0\}$ is in $\Sigma$, has finite measure, and we have $\phi(S - A) = |\phi(S - A) - \int_{S-A}\phi_{n_\varepsilon} d\mu| < \varepsilon$. Since $\phi_n - \mu \to 0$ we may choose $n_1 > n_\varepsilon$ and $B \in \Sigma$ such that $\mu(S - B) < \delta$ and $\phi_{n_1}(s) < \varepsilon/(1 + \mu(A))$ for $s \in B$. Since $n_1 > n_\varepsilon$ we have $|\phi(AB) - \int_{AB}\phi_{n_1} d\mu| < \varepsilon$; hence

$$\phi(AB) \leq \varepsilon + \int_{AB}\phi_{n_1} d\mu < \varepsilon + \varepsilon\mu(AB)/(1 + \mu(A)) < 2\varepsilon.$$

Also $\mu(A - B) \leq \mu(S - B) < \delta$ so that $\phi(A - B) < \varepsilon$. We have

$$\phi(S) = \phi(S - A) + \phi(AB) + \phi(A - B) < 4\varepsilon$$

and since $\varepsilon > 0$ is arbitrary it follows that $\phi(S) = 0$.

Definition 6. A function $f:S \to P_0(X)$ is integrable if there is a sequence $(f_n)$ in $\Gamma$ such that for every continuous seminorm $p$,

(1) $D_p(f_n,f) - \mu \to 0$

(2) $\lim_{n,m\to\infty}\int_S D_p(f_n,f_m)d\mu = 0$, and

(3) $\lim_{n\to\infty}\int_E f_n d\mu$ exists for each $E \in \Sigma$.

The integral of $f$ over $E$ is given by $\int_E f d\mu = \lim_n \int_E f_n d\mu$. The sequence $(f_n)$ is called a determining sequence for $f$ and the set of integrable functions is denoted by $L(\bar{X})$. If $f(s)$ is a real number for all $s$ then the integral is the Lebesgue integral. If $f(s)$ is a single vector of a Banach space for all $s$, then the integral is the Bochner integral.

Remarks: If $P_0(X)$ is complete, then (3) of definition 6 is superfluous by Lemma 4. By Lemma 5 the integral is independent of any particular

determining sequence used. Finally, it follows from Theorem 1.8 that the integral is convex valued (has convex closure).

Lemma 7. For $E \in \Sigma$ fixed, the map $f \to \int_E f d\mu$ on $L(\tilde{X})$ is additive and homogenous.

Proof: If $(f_n)$ and $(g_n)$ are determining sequences for $f,g \in L(\tilde{X})$ respectively and $\alpha$ scalar, then by Lemma 1.30, $D_p(f_n + g_n, f + g) - \mu \to 0$, and $\int_S D_p(f_n + g_n, f_m + g_m)d\mu \to 0$ as $n,m \to \infty$. Furthermore $\int_E (f_n + g_n)d\mu = \int_E f_n d\mu + \int_E f_n d\mu \to \int_E f d\mu + \int_E g d\mu$. Thus $f + g$ is integrable with determining sequence $(f_n + g_n)$ and $\int_E (f + g)d\mu = \int_E f d\mu + \int_E g d\mu$. Similarly $\alpha f \in L(X^-)$ and $\int_E \alpha f d\mu = \alpha \int_E f d\mu$.

Theorem 8. For $f \in L(\tilde{X})$ there is a sequence $(f_n)$ in S such that $\int_S D_p(f_n, f)d\mu \to 0$ for every continuous seminorm p.

Proof: Let $(f_n)$ be a determining sequence for f, p a continuous seminorm. Then $D_p(f_n, f) - \mu \to 0$ and $\int_S D_p(f_n, f_m)d\mu \to 0$. For each $s \in S$ we have

(1) $|D_p(f_m(s), f_n(s)) - D_p(f(s), f_n(s))| \leq D_p(f_m(s), f(s))$ and

(2) $|D_p(f_{m_1}(s), f_n(s)) - D_p(f_{m_2}(s), f_n(s), f_n(s))| \leq D_p(f_{m_1}(s), f_{m_2}(s))$.

From (1) it follows that $D_p(f_m, f_n) - \mu \to D_p(f, f_n)$ as $m \to \infty$ and from (2) we have $\int_S |D_p(f_{m_1}, f_n) - D_p(f_{m_2}, f_n)|d\mu \to 0$, as $m_1, m_2 \to \infty$.

Thus $D_p(f,f_n)$ is integrable with determining sequence $(D_p(f_m,f_n))_m$ and

$$\int_S D_p(f,f_n)d\mu = \lim_m \int_S D_p(f_m,f_n)d\mu.$$

Now for $\varepsilon > 0$ choose $n_\varepsilon$ such that $\int_S D_p(f_m,f_n)d\mu < \varepsilon$ for $n,m \geq m_\varepsilon$. Then $\int_S D_p(f,f_n)d\mu = \lim_m \int_S D_p(f_m,f_n)d\mu \leq \varepsilon$ for $n \geq n_\varepsilon$.

Remarks: The sequence $(f_n)$ in Theorem 8 is the defining sequence for f and we have $D_p(f_n,f) \to \mu \to 0$ for every p. If $f,f_1,f_2,\ldots,\varepsilon L(\tilde{X})$ such that $\lim_n \int_S D_p(f_n,f)d\mu = 0$ for every continuous seminorm p, then we say the sequence $(f_n)$ converges to f in $L(\tilde{X})$ and we write $f_n \to f$ in $L(\tilde{X})$.

Lemma 9. For $f,g \in L(\tilde{X})$, $D_p(f,g)$ is integrable and $D_p(\int_E f d\mu, \int_E g d\mu) \leq \int_E D_p(f,g)d\mu$ for all $E \in \Sigma$.

Proof: Let $(f_n)$, $(g_n)$ be determining sequences for f and g, respectively. For each s we have

$$|D_p(f_n(s),g_n(s)) - D_p(f(s),g(s))| \leq D_p(f_n(s),f(s)) + D_p(g_n(s),g_m(s))$$

and

$$|D_p(f_n(s),g_n(s)) - D_p(f_m(s),g_m(s))| \leq D_p(f_n(s),f_m(s)) + D_p(g_n(s),g(s));$$

hence $D_p(f,g)$ is integrable with determining sequence $(D_p(f_n,g_n))$. It follows from the proof of Theorem 8 that $\int_S |D_p(f_n,g_n) - D_p(f,g)|d\mu \to 0$. By Lemma 4 and Lemma 1.30 we have for each n,

$$D_p\left(\int_E f d\mu, \int_E g d\mu\right)$$

$$\leq D_p\left(\int_E f d\mu, \int_E f_n d\mu\right) + D_p\left(\int_E f_n d\mu, \int_E g_n d\mu\right) + D_p\left(\int_E g_n d\mu, \int_E g d\mu\right)$$

$$\leq r_n + \int_E D_p(f_n, g_n) d\mu$$

where $r_n = D_p\left(\int_E f d\mu, \int_E f_n d\mu\right) + D_p\left(\int_E g_n d\mu, \int_E g d\mu\right)$. Since $r_n \to 0$ and $\int_E D_p(f_n, g_n) d\mu \to \int_E D_p(f, g) d\mu$. the conclusion now follows.

Corollary 10. For $f \in L(\tilde{X})$, $p(f) \in L_1$ and $p\left(\int_E f d\mu\right) \leq \int_E p(f) d\mu$ (put $g = 0$).

Remark. It follows from the proof of the lemma that if $(f_n)$ is a determining sequence for $f$, then $(p(f_n))$ is a determining sequence for $p(f)$.

Lemma 11. Let $f \in L(\tilde{X})$, $\lambda(E) = \int_E f d\mu$. Then $\lambda$ is additive on $\Sigma$ and $v_p(\lambda, E) = \int_E p(f) d\mu$.

Proof: Let $E, F$ be disjoint in $\Sigma$, and let $(f_n)$ be a sequence in $\Gamma$ such that $\int_S D_p(f_n, f) d\mu \to 0$. Since

$$D_p\left(\int_E f_n d\mu + \int_F f_n d\mu, \int_E f d\mu + \int_F f d\mu\right)$$

$$\leq D_p\left(\int_E f_n d\mu, \int_E f d\mu\right) + D_p\left(\int_F f_n d\mu, \int_F f d\mu\right)$$

Lemma 3 yields

$$\lambda(E \cup F) = \int_{E\cup F} f d\mu = \lim_n \int_{E\cup F} f_n d\mu = \lim_n \left(\int_E f_n d\mu + \int_F f_n d\mu\right)$$

$$= \lim_n \int_E f_n d\mu + \lim_n \int_F f_n d\mu = \int_E f d\mu + \int_F f d\mu$$

$$= \lambda(E) + \lambda(F).$$

Next, for $(E_i)$ finite, disjoint in $\Sigma$ with $E_i \subset E$ we have

$$\Sigma_i p(\lambda(E_i)) \le \Sigma_i \int_{E_i} p(f) d\mu = \int_{\cup_i E_i} p(f) d\mu \le \int_E p(f) d\mu$$

and thus $v_p(\lambda,E) = \sup_{(E_i)} \Sigma p(\lambda(E_i)) \le \int_E p(f) d\mu$. Hence for $v_p(\lambda,E) = \infty$ we have $v_p(\lambda,E) = \int_E p(f) d\mu$. Assume $v_p(\lambda,E) < \infty$. Choose $n_\varepsilon$ such that $\int_S D_p(f_n,f) d\mu < \varepsilon$ for $n \ge n_\varepsilon$. Fix $n \ge n_\varepsilon$ and choose finite disjoint sequences $(E_i)$ and $(F_j)$ in $E$ such that $\Sigma_i p(\int_{E_i} f d\mu) > v_p(\lambda,E) - \varepsilon$ and $\Sigma_j p(\int_{F_j} f_n d\mu) > \int_E p(f_n) d\mu - \varepsilon$. Since $\Sigma_i p(\int_{E_i} f d\mu) = \Sigma_i p(\Sigma_j \int_{E_i F_j} f d\mu) \le \Sigma_{ij} p(\int_{E_i F_j} f d\mu) \le v_p(\lambda,E)$ we have

$$|\Sigma_{ij} p(\textstyle\int_{E_i F_j} f d\mu) - v_p(\lambda,E)| \le \varepsilon.$$

Similarly $|\Sigma_{ij} p(\int_{E_i F_j} f_n d\mu) - \int_E p(f_n) d\mu| \le \varepsilon$. Also

$$\begin{aligned} |\Sigma_{ij} p(\textstyle\int_{E_i F_j} f d\mu) - \Sigma_{ij} p(\int_{E_i F_j} f_n d\mu)| &\le \Sigma_{ij} |p(\textstyle\int_{E_i F_j} f d\mu) - p(\int_{E_i F_j} f_n d\mu)| \\ &\le \Sigma_{ij} D_p(\textstyle\int_{E_i F_j} f d\mu, \int_{E_i F_j} f_n d\mu) \\ &\le \Sigma_{ij} \int_{E_i F_j} D_p(f,f_n) d\mu \\ &\le \int_E D_p(f,f_n) d\mu \\ &< \varepsilon; \end{aligned}$$

hence

$$|v_p(\lambda,E) - \int_E p(f_n)d\mu| \leq |v_p(\lambda,E) - \Sigma_{ij} p(\int_{E_i F_j} f d\mu)|$$

$$+ |\Sigma_{ij} p(\int_{E_i F_j} f d\mu) - \Sigma_{ij} p(\int_{E_i F_j} f_n d\mu)|$$

$$+ |\Sigma_{ij} p(\int_{E_i F_j} f_n d\mu) - \int_E p(f_n)d\mu|$$

$$< 3\varepsilon$$

and we have $v_p(\lambda,E) = \lim_n \int_E p(f_n)d\mu = \int_E p(f)d\mu$.

Theorem 12. For f integrable we have (1) $\lim_{\mu(E)\to 0} \int_E f d\mu = 0$, and (2) for each p and $\varepsilon > 0$ there is a set $E_{p\varepsilon} \in \Sigma$ such that $\mu(E_{p\varepsilon}) < \infty$ and $\int_{S-E_{p\varepsilon}} p(f)d\mu < \varepsilon$.

Proof: (1) follows from the same property of the Lebesgue integral for the function p(f) and Corollary 10. For (2) let $\varepsilon > 0$ and choose a simple integrable function $f_\varepsilon$ such that $\int_S D_p(f,f_\varepsilon)d\mu < \varepsilon$. Let $E_{p\varepsilon} = \{f_\varepsilon^{-1}(A): A \neq \{0\}\}$. Since $f_\varepsilon$ is simple we have $\mu(E_{p\varepsilon}) < \infty$ and

$$\int_{S-E_{p\varepsilon}} p(f)d\mu \leq \int_{S-E_{p\varepsilon}} D_p(f,f_\varepsilon)d\mu + \int_{S-E_{p\varepsilon}} p(f_\varepsilon)d\mu < \varepsilon + 0.$$

Theorem 13. If $\mu$ is countably additive and f is integrable, then $E \to \int_E f d\mu$ is countably additive and convex valued almost everywhere.

Proof: Let $(E_n)$ be an increasing sequence in $\Sigma$ with union $E \in \Sigma$. Since $p(f) \in L_1$, $\int p(f)d\mu$ is a finite measure; hence $\int_{E-E_n} p(f)d\mu \to 0$. Thus for every p we have

$$\begin{aligned} D_p\left(\int_{E_n} f d\mu, \int_E f d\mu\right) &= D_p\left(\int_E 1_{E_n} f d\mu, \int_E f d\mu\right) \le \int_E D_p(1_{E_n} f, f) d\mu \\ &= \int_{E-E_n} D_p(1_{E_n} f, f) d\mu + \int_{E_n} D_p(1_{E_n} f, f) d\mu \\ &= \int_{E-E_n} p(f) d\mu \to 0. \end{aligned}$$

To show f is almost every convex valued let $(f_n)$ be a determining sequence for f. Then $D_p(f_n, f) - \mu \to 0$ and by Egoroff's theorem there is a subsequence $(n_k)$ such that $D_p(f_{n_k}, f) \to 0$ almost everywhere. The conclusion now follows from Theorem 1.9.

Theorem 14 (Vitali Convergence Theorem). Let $(f_n)$ be a sequence in $L(\bar{X})$, $f:S \to P(X)$. Then f is integrable and $f_n \to f$ in $L(\bar{X})$ if and only if for every continuous seminorm p,

(1) $D_p(f_n, f) - \mu \to 0$,

(2) $\lim_{\mu(E)\to 0} \int_E p(f_n) d\mu = 0$ uniformly for $n = 1,2,\ldots$, and

(3) for each $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ such that $\mu(E_\varepsilon) < \varepsilon$ and $\int_{S-E_\varepsilon} p(f_n) d\mu < \varepsilon$ for all n.

The proof is exactly the one found in Theorem III.3.6 of Dunford and Schwartz [Du58] with p=1 and the Hausdorff p-distance replacing the ordinary distance.

Theorem 15 (Dominated Convergence Theorem). Let $f_n, g$ be integrable, $n = 1,2,\ldots$, such that for every continuous p, $p(f_n) \leq p(g)$ almost everywhere. Then a function f is integrable and $f_n \to f$ in $L(\bar{X})$ if and only if $D_p(f_n,f) \to \mu \to 0$ for every p.

The proof is a direct consequence of Theorem 14.

## III.4 Comparison With Other Integrals

In this section it is assumed that X is a Banach space and will be written as $\Xi$

Definition 1. A function $f:S \to P(\Xi)$ is Aumann integrable if there is a function $\phi \in L_1(S,\Sigma,\mu,\Xi)$ such that $\phi(s) \in f(s)$ except possibly on a null set. The Aumann integral of f over a set E is given by $(A)\int_E f d\mu = \{\int_E \phi d\mu : \phi \in L_1(S,\Sigma,\mu,\Xi), \phi \in f \text{ a.e.}\}$. The set of Aumann integrable functions will be denoted by $A(\bar{\Xi})$ and the functions $\phi$ will be called integrable selections.

First we state a counterpart of Lemma 3.9 for the Hausdorff semidistance.

Lemma 2. For $f,g \in L(\bar{\Xi})$ $\delta(f,g)$ is an integrable scalar function such that $\delta(\int_E f d\mu, \int_E g d\mu) \leq \int_E \delta(f,g) d\mu$ for all $E \in \Sigma$.

Lemma 3. Let $\Xi$ be a reflexive space, $f_n, f \in L(\bar{\Xi})$ such that $f_n \to f$ in $L(\bar{\Xi})$, and $\phi_n$ an integrable selection of $f_n$ for each n. Then $\{\phi_n\}_n$ is relatively weakly compact in $L_1(S,\Sigma,\mu,\Xi)$.

Proof: Let $\varepsilon > 0$ and choose $n_\varepsilon$ such that $\int_S D(f_n,f)d\mu < \varepsilon$ for $n \geq n_\varepsilon$. Then for $n \geq n_\varepsilon$, $\int_S |\phi_n| d\mu \leq \int_S |f_n| d\mu \leq \int_S D(f_n,f)d\mu + \int_S |f| d\mu < \varepsilon + \int_S |f| d\mu$ so $\{\phi_n\}_n$ is bounded in $L_1$. Since $\int_E |\phi_n| d\mu \leq \varepsilon + \int_E |f| d\mu$ for $n \geq n_\varepsilon$ it follows that $\lim_{\mu(E)\to 0} \int_E |\phi_n| d\mu = 0$ uniformly for $n = 1,2,\ldots$, and since $\mu(S) < \infty$, the sequence $\{\int |\phi_n| d\mu\}_n$ of measures is uniformly strongly additive. Thus $\{\phi_n\}$ is relatively weakly compact by Theorem 1 of Brooks [Br72].

Theorem 4. If $\mu$ is countably additive and $\Xi$ is reflexive, then for every $f \in L(\bar{\Xi})$, we have $f \in A(\Xi)$ and $\int_E f d\mu = (A)\int_E f d\mu$ for all $E \in \Sigma$.

Proof (Debrew [De65]): Let $(f_n)$ be a determining sequence for $f$, and for each $n$ let $\phi_n$ be a simple integrable selection of $f_n$. By lemma 3 we may assume without loss of generality that the sequence $(\phi_n)$ converges weakly in $L_1(S,\Sigma,\mu,\Xi)$ to a function $\phi$. If it can be shown that $\phi$ is an integrable selection of $f$, then it follows that $f \in A(\Xi)$. Let $\varepsilon > 0$ and choose $n_\varepsilon$ such that $\int_S D(f_n,f)d\mu < \varepsilon$ for $n \geq n_\varepsilon$. Since the sequence $(\phi_n)_{n=n_\varepsilon}$ converges weakly to $f$, there is by a corollary of Mazur's theorem (see Corollary V.3.14 of Dunford and Schwartz [Du58]) a sequence of convex combinations converging strongly to $\phi$. In particular there is a convex combination $\phi_\varepsilon = \Sigma_i \lambda_i \phi_{n_i}, n_i > n_\varepsilon$, such that $\int_S |\phi - \phi_\varepsilon| d\mu < \varepsilon$. Let $s \in S$ such that $f(s)$ is convex. (Recall that $f$ is almost everywhere convex valued by Theorem 13.) Then

$$\delta(\phi_\varepsilon(s),f(s)) \leq \Sigma_i \lambda_i \delta(\phi_{n_i}(s),f(s)).$$

In fact, for $r > \Sigma_i \lambda_i \delta(\phi_{n_i}(s), f(s))$, there are numbers $r_i > \lambda_i \delta(\phi_{n_i}(s), f(s))$ such that $r = \Sigma_i r_i$. Then $\delta(\phi_{n_i}(s), f(s)) < r_i/\lambda_i$ so that $\phi_{n_i}(s) \in f(s) + (r_i/\lambda_i)S_1$. Choose $z_i \in f(s)$ such that $|\phi_{n_i}(s) - z_i| < r_i/\lambda_i$. Since $f(s)$ is convex, for $z = \Sigma \lambda_i z_i$ we have $z \in f(s)$ and

$$|\phi_\varepsilon(s) - z| = |\Sigma_i \lambda_i \phi_{n_i}(s) - \Sigma_i \lambda_i z_i| \leq \Sigma_i \lambda_i |\phi_{n_i}(s) - z_i| \leq \Sigma_i r_i = r.$$

Then $\phi_\varepsilon(s) \in f(s) + rS_1$ hence $\delta(\phi_\varepsilon(s), f(s)) < r$. It now follows that

$$\int_S \delta(\phi_\varepsilon, f) d\mu \leq \Sigma_i \lambda_i \int_S \delta(\phi_{n_i}, f) d\mu \leq \Sigma_i \lambda_i \int_S D(\phi_{n_i}, f) d\mu \leq \Sigma_i \lambda_i \varepsilon = \varepsilon.$$

Since $\phi$, $\phi_\varepsilon$ are single valued, we have $\int_S \delta(\phi, \phi_\varepsilon) d\mu = \int_S |\phi - \phi_\varepsilon| d\mu < \varepsilon$. Thus $\int_S \delta(\phi, f) d\mu \leq \int_S \delta(\phi, \phi_\varepsilon) d\mu + \int_S \delta(\phi_\varepsilon, f) d\mu < 2\varepsilon$ and since $\varepsilon > 0$ is arbitrary we have shown that $\int_S \delta(\phi, f) d\mu = 0$. Since the function $\delta(\phi(s), f(s))$ is nonnegative, it follows that $\delta(\phi, f) = 0$ almost everywhere. By Lemma 1.34 $\phi(s)$ is contained in the closure of $f(s)$ for almost all $s$ and thus $\phi$ is an integrable selection of $f$.

Now, let $E \in \Sigma$, and $x \in \int_E f d\mu$. Then $D(\int_E f_n d\mu, \int_E f d\mu) \to 0$ and by Lemma 1.31 there is a selection $(x_n)$ of $(\int_E f_n d\mu)_n$ such that $x_n \to x$. The simple function $f_n$ has the form $f_n = \Sigma_i 1_{E_{ni}} X_{ni}$ and thus $x_n$ has the form $x_n = \Sigma_i \mu(EE_{ni}) x_{ni}$ where $x_{ni} \in X_{ni}$. If $\phi_n = \Sigma_i 1_{E_{ni}} x_{ni}$, then $\phi_n$ is a simple integrable selection of $f_n$ such that $x_n = \int_E \phi_n d\mu$. We may assume $(\phi_n)$ is weakly convergent to an integrable selection $\phi$ of $f$.

Since $\psi \to x^* \int_E \psi d\mu$ is a continuous linear functional on $L_1(S,\Sigma,\mu,\Xi)$, it follows that $x^* x_n = x^* \int_E \phi_n d\mu \to x^* \int_E \phi d\mu$ and, therefore, $x = \int_E \phi d\mu$; hence $x \in (A)\int_E f d\mu$. It remains to show that $(A)\int_E f d\mu \subset \int_E f d\mu$. Let $x \in (A)\int_E f d\mu$. Then $x = \int_E \phi d\mu$ for some integrable selection $\phi$ of $f$. Since $\phi(s) \in f(s)$ almost everywhere we have $\delta(\phi,f) = 0$ almost everywhere hence $\delta(\int_E \phi d\mu, \int_E f d\mu) \leq \int_E \delta(\phi,f) d\mu = 0$ thus $x = \int_E \phi d\mu \in \int_E f d\mu$.

Remarks. It is easy to show that $A(\bar{\Xi})$ is generally not contained in $L(\Xi^-)$. Indeed, let $S = (0,1]$ with Lebesgue measure $\mu$ and consider $f:S \to P_0(R)$ defined by $f(s) = \{0,1\}$ for all $s$. Then $f$ is not in $L(\Xi^-)$ by Theorem 13. However, for any $x$ in $[0,1]$, if we let $\phi_x(s) = 1_{[0,x]}$ then $\phi_x$ is an integrable selection of $f$ and therefore $f$ is Aumann integrable. In fact since $x$ is arbitrary and $x = (A)\int_S \phi_x d\mu \in (A)\int_S f d\mu$ we have $(A)\int_S f d\mu = [0,1]$. If $A_c(\bar{\Xi})$ denotes those elements of $A(\bar{\Xi})$ which are almost everywhere convex, then it is still not true that $A_c(\bar{\Xi})$ is contained in $L(\bar{\Xi})$ generally.

Let $f:S \to P_0(R)$ defined by $f(s) = [0,1/s^2]$. We will show that the Aumann integral of $f$ exists but is not absolutely continuous with respect to $\mu$. Then it follows from Theorem 3.12 that $f$ is not in $L(\Xi^-)$. Let $E_n = (0,1/n]$ for $n=1,2,\ldots$, so that $\mu(E_n) \to 0$. Fix $n$ and let $r \geq n$ be an arbitrary real number. Define $\phi_r(s) = 0$ for $s \in (1/r,1]$ and $\phi_r(s) = r^2$ for $s \in (0,1/r]$. Then $\phi_r$ is Lebesgue integrable and $\int_{E_n} \phi_r d\mu = r^2 \mu(0,1/r] = r$. If $s \in (1/r,1]$ then $\phi_r(s) = 0 \in [0,1/s^2] = f(s)$ and if $s \in (0,1/r]$, then $r^2 \leq 1/s^2$

thus $\phi_r(s) = r^2 \in [0,1/s^2] = f(s)$; hence $\phi_r$ is an integrable selection of f. It follows that $f \in A(\tilde{\Xi})$ and that $r = \int_{E_n} \phi_r d\mu \in (A)\int_{E_n} f d\mu$. Since $r \geq n$ is arbitrary it follows that $[n,\infty) \subset (A)\int_{E_n} f d\mu$ and thus $(A)\int_{E_n} f d\mu \not\to 0$. Note that since 0 is clearly in the Aumann integral over $E_n$ and the Aumann integral is convex (see Theorem 1 of Aumann [Au65]) it follows that $(A)\int_{E_n} f d\mu = R^+$ for each n. This example also shows that the Aumann integral is not countably additive since $E_n$ is a decreasing sequence with void intersection in a finite measure space.

Theorem 5 (Radstrom). Let K denote the compact convex subsets of a real Banach space $\Xi$. Then K can be isometrically embedded as a closed convex cone in a real Banach space $\Xi'$. The image of a set $\kappa \in K$ in $\Xi'$ will be denoted by $\kappa'$. The metric on K is the Hausdorff metric and $D(\kappa_1,\kappa_2) = |\kappa_1' - \kappa_1'|$.

Definition 6. A function $f:S \to K$ is Debrew integrable if the associated function $f'(s) = (f(s))'$ is in $L_1(S,\Sigma,\mu,\Xi')$, that is, if there is a sequence $(\psi_n)$ of simple integrable $\Xi'$-valued functions such that $\psi_n - \mu \to f'$ and $\int_S |\psi_n - \psi_m| d\mu \to 0$. The Debrew integral of f is the ordinary (Bochner) integral of $f'$. By Theorem 6.1 of Debrew [De65] there is a sequence $(f_n)$ of simple integrable K-valued functions such that $f_n' - \mu \to f'$ and $\int_S |f_n' - f_m'| d\mu \to 0$. By the isometry we have $D(f_n,f) - \mu \to 0$ and $\int_S D(f_n,f_m) d\mu \to 0$ thus $f \in L(\tilde{\Xi})$ with $\int_E f d\mu = \lim_n \int_E f_n d\mu$ (since $\Xi$ is a complete metric space and therefore $P_0(\Xi)$ is complete for the Hausdorff metric). The Debrew

integral of f is $\int_E f' d\mu = \lim_n \int_E f_n' d\mu = (\lim_n \int_E f_n d\mu)' = (\int_E f d\mu)'$. Thus our integral includes the Debrew integral as a special case.

## III.5 The Weak Integral

As before, $\Xi$ denotes a Banach space, $\bar{\Xi}$ the class of bounded convex subsets of $\Xi$, and elements of $\Xi$ are considered equal if they have the same closure. $\Xi^*$ denotes the dual of $\Xi$.

Lemma 1. If f is integrable, then for a bounded linear operator u on $\Xi$, the function uf is integrable and $\int_E uf d\mu = u\int_E f d\mu$.

Proof: For f simple so is the result. In general, let $(f_n)$ be a determining sequence for f. Since $D(uf_n(s), uf(s)) \leq |u| D(f_n(s), f(s))$ by Lemma 1.29a it follows that $D(uf_n, uf) - \mu \to 0$ and, similarly,

$$\int_S D(uf_n, uf_m) d\mu \leq |u| \int_S D(f_n, f_m) d\mu \to 0;$$

hence uf is integrable with determining sequence $(uf_n)$. We have

$$\int_E uf d\mu = \lim_n \int_E uf_n d\mu = \lim_n u \int_E f_n d\mu = u(\lim_n \int_E f_n d\mu) = u\int_E f d\mu.$$

Definition 2. A function $f: S \to P_0(\Xi)$ is weakly integrable if $x^* f$ is integrable for each $x^* \in \Xi^*$ and if for each $E \in \Sigma$ there is a convex set $X_{Ef}$ in $\Xi$ such that $x^* X_{Ef} = \int_E x^* f d\mu$ for each $x^*$. The set $X_{Ef}$ is called the weak integral of f over E and is denoted by $(w)\int_E f d\mu$.

Lemma 3. $f \to (w)\int_E f d\mu$ is additive and homogenous on the class of weakly integrable functions.

Proof: Let f,g be weakly integrable, $E \in \Sigma$, $X = (w)\int_E f d\mu + (w)\int_E g d\mu$. Then X is convex and

$$x^*X = x^*X_{Ef} + x^*X_{Eg} = \int_E x^* f d\mu + \int_E x^* g d\mu = \int_E x^*(f + g) d\mu$$

by Lemma 3.7. Thus f + g is weakly integrable and $(w)\int_E (f + g) d\mu = X$. Similarly, for $\alpha$ scalar, $\alpha f$ is weakly integrable and $(w)\int_E \alpha f d\mu = \alpha(w)\int_E f d\mu$.

Lemma 4. If f is weakly integrable, then the function $E \to (w)\int_E f d\mu$ is additive on $\Sigma$.

Proof: Let E,F be disjoint in $\Sigma$, $x^* \in \Xi^*$. Since the strong integral is additive we have

$$x^*(w)\int_{E \cup F} f d\mu = \int_{E \cup F} x^* f d\mu = \int_E x^* f d\mu + \int_F x^* f d\mu$$

$$= x^*(w)\int_E f d\mu + x^*(w)\int_F f d\mu = x^*((w)\int_E f d\mu + (w)\int_F f d\mu).$$

Since $(w)\int_E f d\mu + (w)\int_F f d\mu$ and $(w)\int_{E \cup F} f d\mu$ are convex sets, they are equal by Lemma 1.37.

Theorem 5. For f weakly integrable and $\mu$ countably additive, $E \to (w)\int_E f d\mu$ is countably additive.

Proof: The result follows from Theorem 2.14 and Theorem 3.13.

Theorem 6. Assume $\mu(s) < \infty$ and let $(f_n)$ be a sequence of weakly integrable functions such that the limit $\lim_n (w)\int_E f_n d\mu$ exists for each

$E \in \Sigma$. If $f:S \to P_0(\Xi)$ is a function such that $D(x^*f_n, x^*f) - \mu \to 0$ for each $x^*$, then f is weakly integrable and $(w)\int_E f d\mu = \lim_n (w)\int_E f_n d\mu$.

Proof: Let $x^* \in \Xi^*$. Then $x^*f_n \in L(\bar{\Xi})$ for $n = 1,2,\ldots$, and $\lim_n \int_E x^* f_n d\mu$ exists by Lemma 1.29a. By Theorem 3.12, $\int x^* f d\mu$ is strongly additive since $\mu$ is finite. By Theorem 2.8 we have $\lim_{\mu(E)\to 0} \int_E x^* f_n d\mu = 0$ uniformly for $n = 1,2,\ldots$. It follows from Lemma 3.11 and Theorem 2.18 that $\lim_{\mu(E)\to 0} \int_E |x^* f_n| d\mu = 0$ uniformly. By the Vitali Convergence theorem for $L(\bar{\Xi})$, $x^*f \in L(\bar{\Xi})$ and $x^*f_n \to x^*f$ in $L(\bar{\Xi})$. Thus

$$x^* \lim_n (w)\int_E f_n d\mu = \lim_n x^*(w)\int_E f_n d\mu$$

$$= \lim_n \int_E x^* f_n d\mu = x^*(w)\int_E f d\mu$$

and therefore f is weakly integrable and $(w)\int_E f d\mu = \lim_n (w)\int_E f_n d\mu$.

### III.6 Enlarging the Class of Integrable Functions

Lemma 1. Let p be a continuous seminorm on X. If $(f_n)$ is a sequence in $L(\bar{\Xi})$ such that

(1) $D_p(f_n, f_m) - \mu \to 0$,

(2) $\lim_{\mu(E)\to 0} \int_E p(f_n) d\mu = 0$ uniformly in n, and

(3) given $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ such that $\mu(E_\varepsilon) < \infty$ and $p(\int_E f_n d\mu) < \varepsilon$ for all n and $E \in \Sigma$ with $E \subset S - E_\varepsilon$,

then $\lim_{n,m} D_p(\int_E f_n d\mu, \int_E f_m d\mu) = 0$ uniformly for $E \in \Sigma$.

Proof: Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\int_E p(f_n)d\mu < \varepsilon$ for all $n$ and $E$ with $\mu(E) < \delta$. For $n,m$ large there is by (1) a set $F \in \Sigma$ such that $\mu(F) < \delta$ and $D_p(f_n(s),f_m(s)) < \varepsilon/(1 + \mu(E_\varepsilon))$ for $s \in S - F$. Hence

$$\begin{aligned}
D_p\Big(\int_E f_n d\mu, \int_E f_m d\mu\Big) &= D_p\Big(\int_{E-E_\varepsilon} f_n d\mu + \int_{EE_\varepsilon F} f_n d\mu + \int_{EE_\varepsilon - F} f_n d\mu, \\
&\qquad \int_{E-E_\varepsilon} f_m d\mu + \int_{EE_\varepsilon F} f_m d\mu + \int_{EE_\varepsilon - F} f_n d\mu\Big) \\
&\le D_p\Big(\int_{E-E_\varepsilon} f_n d\mu, \int_{E-E_\varepsilon} f_m d\mu\Big) + D_p\Big(\int_{EE_\varepsilon F} f_n d\mu, \int_{EE_\varepsilon F} f_m d\mu\Big) \\
&\quad + D_p\Big(\int_{EE_\varepsilon - F} f_n d\mu, \int_{EE_\varepsilon - F} f_m d\mu\Big) \\
&\le p\Big(\int_{E-E_\varepsilon} f_n d\mu\Big) + p\Big(\int_{E-E_\varepsilon} f_m d\mu\Big) \\
&\quad + p\Big(\int_{EE_\varepsilon F} f_n d\mu\Big) + p\Big(\int_{EE_\varepsilon F} f_m d\mu\Big) + \int_{EE_\varepsilon - F} D_p(f_n,f_m)d\mu \\
&< 5\varepsilon.
\end{aligned}$$

Definition 2. Assume $P_0(X)$ is complete (if $X$ is a Banach space for example). A function $f:S \to P_0(X)$ will now be called integrable if there is a sequence $(f_n)$ in $L(\bar{X})$ such that for every continuous seminorm $p$,

(1) $D_p(f_n,f) - \mu \to 0$,

(2) $\lim_{\mu(E)\to 0}\int_E p(f_n)d\mu = 0$ uniformly in $n$, and

(3) for every $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ with finite measure such that $p(\int_E f_n d\mu) < \varepsilon$ for all $n$ and $E \in \Sigma$ with $E \subset S - E_\varepsilon$.

Since $P_0(X)$ is complete Lemma 1 shows that the limit $\lim_n \int_E f_n d\mu$ exists for every $E \in \Sigma$. The integral of f over E is given by $\int_E f d\mu = \lim_n \int_E f_n d\mu$. It follows from the proof of Lemma 1 that $\lim_n \int_E g_n d\mu = \lim \int_E f_n d\mu$ for any other sequence $(g_n)$ also satisfying conditions (1), (2), and (3).

Remarks. In this section functions in $L(\tilde{X})$ will be called strongly integrable. It is clear that any strongly integrable function f is integrable in the new sense by taking the sequence $(f_n)$ where $f_n = f$ for all n.

Theorem 3. If X is normed, then for f integrable there is a sequence $(g_n)$ of simple (strongly) integrable functions satisfying the conditions of definition 2. Thus f is strongly integrable by the Vitali Convergence Theorem.

Proof: Let $(f_n)$ be a sequence from definition 2. The topology of convergence in measure for real valued functions on S is a metric topology (see Lemma III.2.7 of Dunford and Schwartz [Du58]) with norm $|r|_F$. Since $f_n \in L(\tilde{X})$ there is a sequence of simple functions converging in measure to $f_n$ and also in $L(\tilde{X})$. Therefore there is a simple function $g_n$ such that $|D(f_n,g_n)|_F < 1/n$ and $\int_S D(f_n,g_n)d\mu < 1/n$. Then $|D(f,g_n)|_F \leq |D(f,f_n)|_F + 1/n \to 0$; hence $D(g_n,f) - \mu \to 0$. Let $\varepsilon > 0$ and choose $n_\varepsilon$ such that $1/n_\varepsilon < \varepsilon$. Choose $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f_n| d\mu < \varepsilon$ for all n and $\int_E |g_n| d\mu < \varepsilon$ for

$n = 1,2,\ldots,n_\varepsilon$. Then for $\mu(E) < \delta$ and $n \geq n_\varepsilon$ we have $\int_E |g_n| d\mu \leq \int_E D(g_n, f_n) d\mu + \int_E |f_n| d\mu < 1/n + \varepsilon \leq 1/n_\varepsilon + \varepsilon < 2\varepsilon$ and it follows that $\lim_{\mu(E)\to 0} \int_E |g_n| d\mu = 0$ uniformly in $n$. Let $E_\varepsilon$ be the set given by condition (3) and let $E'$ be the union of the sets $E_{g_n} = \{s : g_n(s) \neq 0\}$ for $n = 1,\ldots,n_\varepsilon$. Since $g_n$ is integrable $\mu(E_{g_n}) < \infty$, and for $F_\varepsilon = E_\varepsilon \cup E'$ we have $\mu(F_\varepsilon) < \infty$. Let $E \subset S - F_\varepsilon$ with $E \in \Sigma$. For $n \leq n_\varepsilon$ we have $\left|\int_E g_n d\mu\right| = 0$ and for $n \geq n_\varepsilon$ we have $\left|\int_E g_n d\mu\right| \leq \int_E D(g_n, f_n) d\mu + \left|\int_E f_n d\mu\right| < 1/n_\varepsilon + \varepsilon < 2\varepsilon$. The theorem is proved.

Lemma 4. For $E \in \Sigma$ the function $f \to \int_E f d\mu$ is additive and homogenous on the class of integrable functions.

Lemma 5. For $f$ integrable the set function $E \to \int_E f d\mu$ is additive and $\mu$-continuous on $\Sigma$ and for $\varepsilon > 0$ there is a set $E_\varepsilon \in \Sigma$ such that $\mu(E_\varepsilon) < \infty$ and $p(\int_E f d\mu) < \varepsilon$ for $E \subset S - E_\varepsilon$.

Lemma 6. The integral is convex valued.

Theorem 7. If $\mu$ is countably additive and finite, then $\int f d\mu$ is countably additive.

Theorem 8. Let $f : S \to P_0(X)$ and $(f_n)$ a sequence of integrable (not necessarily in $L(\bar{X})$) functions satisfying the conditions of definition 2. Then $f$ is integrable and $\int_E f d\mu = \lim_n \int_E f_n d\mu$.

Proof: For each n choose a sequence $(f_{nk})$ in $L(\tilde{X})$ such that $D_p(f_{nk},f_n) - \mu \to 0$ and $\int_E f_{nk}d\mu \to \int_E f_n d\mu$, $E \in \Sigma$. Let $\varepsilon > 0$ and fix p. For each n there is an integer $k_n$ such that $D_p(\int_E f_n d\mu, \int_E f_{nk} d\mu) < \varepsilon$ for $k \geq k_n$ and without loss of generality we assume $D_p(\int_E f_n d\mu, \int_E f_{nk} d\mu) < \varepsilon$ for all n,k. Now for each n choose $k_n$ such that $|D_p(f_n,f_{nk})|_F < 1/n$. Then $D_p(g_n,f) - \mu \to 0$ where $g_n = f_{nk}$. Clearly (2) and (3) hold for $(g_n)$. It follows that f is integrable and $\int_E f d\mu = \lim_n \int_E g_n d\mu$. Also for every p and $\varepsilon > 0$ we have

$$D_p(\int_E f d\mu, \int_E f_n d\mu) \leq D_p(\int_E f d\mu, \int_E g_n d\mu) + D_p(\int_E g_n d\mu, \int_E f_n d\mu)$$

$$< \varepsilon \text{ for } n \text{ large.}$$

Theorem 9. Assume $\mu$ is finite. If $(f_n)$ is a sequence of integrable functions and f is a function such that $D_p(f_n,f) - \mu \to 0$ for every p and the limit $\lim_n \int_E f_n d\mu$ exists for every $E \in \Sigma$, then f is integrable and $\int_E f d\mu = \lim_n \int_E f_n d\mu$.

Proof: The result follows from Theorem 8 and Theorem 2.8.

Theorem 10. If $\mu$ is finite and f is integrable, then f is weakly integrable.

Proof: Let $(f_n)$ be a sequence of strongly integrable functions satisfying the conditions of definition 2. Then $(f_n)$ and f satisfy the hypothesis of Theorem 5.6 and the result follows.

Remarks. If $\Xi$ is a Banach space, a weak integral based on the integral of this section and definition 5.2 is easily defined. Finally, another approach to enlarging the class of integrable functions is given as follows: A function f is integrable if there is a sequence $(f_n)$ of simple functions in $L(X^-)$ such that $D_p(f_m,f) \to \mu \to 0$ for every p and $\lim_n \int_E f_n d\mu$ exists for every $E \in \Sigma$.

## III.7 Integration With Respect to a Vector-Set Valued Measure

An integration theory for set valued measures is presented in Brooks [Br68a] and [Br68b]. We establish here a kind of monotone convergence theorem for the integral.

Lemma 1. Let $A,B,A_n,B_n$, $n = 1,2,\ldots$, be bounded subsets of X such that $A_n \subset B_n$ for every n, $D_p(A_n,A) \to 0$, and $D_p(B_n,B) \to 0$. Then $A \subset \bar{B}$.

Proof: Fix $a \in A$ and let p and let $\varepsilon > 0$ be given. Choose $n_1$ such that $D_p(A_{n_1},A) < \varepsilon$ and $D_p(B_{n_1},B) < \varepsilon$ for $n \geq n_1$. Let $x \in A_{n_1}$ such that $p(a - x) < \varepsilon/2$. Since $x \in B_{n_1}$ there is an element $b \in B$ such that $p(x - b) < \varepsilon/2$; hence $p(a - b) < \varepsilon$. It follows that x is in any neighborhood of B.

Definition 2. Let $\mu:\Sigma \to P_0(X)$ be additive. $\mu$ is monotone if $\mu(E) \subset \mu(F)$ for $E,F \in \Sigma$ with $E \subset F$. Note that $\mu(\phi) = \{0\}$ since $\mu$ is additive and thus for a monotone function $\mu$ one has $0 \in \mu(E)$ for all $E \in \Sigma$.

Lemma 2. Let $\mu$ be a scalar valued function on $\Sigma$. A real valued function f on S is the limit in $\mu$-measure of a sequence of simple functions on S if and only if for each $\varepsilon > 0$ there are disjoint sets $E_\varepsilon$, $A_1^\varepsilon$, $A_2^\varepsilon,\ldots,A_n^\varepsilon$ in $\Sigma$ such that $S\backslash E_\varepsilon = \bigcup A_i^\varepsilon$, $v(\mu,E_\varepsilon) < \varepsilon$, and $\sup_{s,t\in A_i^\varepsilon} |f(s) - f(t)| < \varepsilon$.

Proof: Suppose there is a sequence $(f_n)$ of simple functions converging to f in $\mu$-measure. Let $\varepsilon > 0$ and choose $n_\varepsilon$ such that $v^*(\mu,\{s:|f(s) - f_{n_\varepsilon}(s)| > \varepsilon/3\}) < \varepsilon$. Choose a set $E_\varepsilon \in \Sigma$ with $\{s:|f(s) - f_{n_\varepsilon}(s)| > \varepsilon/3\} \subset E_\varepsilon$ and $v(\mu,E_\varepsilon) < \varepsilon$. Let $f_{n_\varepsilon} = \Sigma a_i 1_{E_i}$ where $(E_i)$ is a finite disjoint sequence with $S = \bigcup_i E_i$. For $A_i^\varepsilon = E_i \bigcap (S\backslash E_\varepsilon)$ we have $\bigcup A_i^\varepsilon = S\backslash E_\varepsilon$ and for all $s,t \in A_i^\varepsilon$ we have

$$|f(s) - f(t)| \le |f(s) - f_{n_\varepsilon}(s)| + |f_{n_\varepsilon}(s) - f_{n_\varepsilon}(t)| + |f_{n_\varepsilon}(t) - f(t)|$$

$$< 2\varepsilon/3,$$

thus $\sup_{s,t\in A_i^\varepsilon} |f(s) - f(t)| < \varepsilon$.

Conversely, suppose the sets exist for arbitrary $\varepsilon > 0$. For a positive integer k, let $f_k = \Sigma_{i=1}^{n_k} a_i 1_{F_{ik}}$ where $F_{ik} = A_i^{(1/k)}$ and $a_i = f(s)$ for some $s \in F_{ik}$. Let $\delta > \varepsilon > 0$ and choose $k_\varepsilon$ such that $1/k_\varepsilon < \varepsilon$. Then for $k \ge k_\varepsilon$ we have $\{s:|f_k(s) - f(s)| > \delta\}$

$\{s:|f_k(s) - f(s)| > 1/k_\varepsilon\} \subset E_{1/k_\varepsilon}$ and thus $v^*(\mu,\{s:|f_k(s) - f(s)| > \delta\})$ $\leq v(\mu,E_{1/k_\varepsilon}) < \varepsilon$. It follows that the sequence $(f_k)$ of simple functions converges in measure to f.

Lemma 3. If f is a positive totally $\mu$-measurable function on S, then there is a sequence $(f_n)$ of simple functions converging to f in measure with $0 \leq f_n \leq f$ for all n.

Proof: For each n there are disjoint sets $A_1^n, A_2^n, \ldots, A_{k_n}^n$ in $\Sigma$ such that $v(\mu, S\setminus \bigcup_i A_i^n) < 1/n$ and $\sup_{s,t\in A_i^n} |f(s) - f(t)| < 1/n$. Define $f_n = \Sigma_{i=1}^{k_n} a_i 1_{A_i^n}$ where $a_i = \inf_{s\in A_i^n} f(s)$. Then $f_n$ is a simple function such that $0 \leq f_n(s) \leq f(s)$ for all $s \in S$. If $s \in A_i^n$, then $|f(s) - a_i| \leq 1/n$ so that $\{s:|f_n(s) - f(s)| > 1/n\} \subset S\setminus \bigcup_i A_i^n$. Thus $v^*(\mu,\{s:|f_n(s) - f(s)| > 1/n\}) < 1/n$. It follows that $f_n - \mu \to f$.

Lemma 4. Let $\mu:\Sigma \to \bar{X}$ be an additive monotone function. If f,g are nonnegative integrable functions on S such that $f \leq g$ almost everywhere, then for $E \in \Sigma$, $\int_E f d\mu \subset \int_E g d\mu$.

Proof: For $0 \leq a \leq b$ and $E \in \Sigma$, $0 \in \mu(E)$ and since $\mu(E)$ is convex, $a\mu(E) \subset b\mu(E)$. Suppose f and g are simple. We may write $f = \Sigma a_i 1_{E_i}$, $g = \Sigma b_i 1_{E_i}$ where $a_i \leq b_i$ for all i. Then for $E \in \Sigma$ we have

$$\int_E f d\mu = \Sigma a_i \mu(EE_i) \subset \Sigma b_i (EE_i) = \int_E g d\mu.$$

For general f and g, choose sequences $(f_n)$ and $(h_n)$ of simple functions converging in measure to f and g, respectively, such that $0 \leq f_n \leq f$, and $0 \leq h_n \leq g$ for all n. The function $g_n = f_n \vee h_n$ is a simple function such that $f_n \leq g_n \leq g$ for all n. For each $\varepsilon > 0$ the inclusion $\{s: |g_n(s) - g(s)| > \varepsilon\}$ $\{s: |h_n(s) - g(s)| > \varepsilon\}$ obtains and thus $(g_n)$ converges in measure to g. Since $f_n$ and $g_n$ are simple with $f_n \leq g_n$ we have $\int_E f_n d\mu \subset \int_E g_n d\mu$ for each n. Now since f and g are integrable the dominated convergence theorem of Brooks shows that $\int_E |f_n - f| dv(\mu) \to 0$ and $\int_E |g_n - g| dv(\mu) \to 0$ thus by Lemma 2(ii) of Brooks [Br68a] $D(\int_E f_n d\mu, \int_E f d\mu) \to 0$ and $D(\int_E g_n d\mu, \int_E g d\mu) \to 0$. By Lemma 1 $\int_E f d\mu$ is contained in the closure of $\int_E g d\mu$.

Theorem 5. If $(f_n)$ is an increasing sequence of positive integrable functions converging pointwise to an integrable function f, then the sequence of sets $(\int_E f_n d\mu)_n$ increases to the set $\int_E f d\mu$ for every $E \in \Sigma$.

Proof: By Lemma 4, for each n we have $\int_E f_n d\mu \subset \int_E f_{n+1} d\mu \subset \int_E f d\mu$. By the dominated convergence theorem, $\int_S |f_n - f| dv(\mu) \to 0$, hence $D(\int_E f_n d\mu, \int_E f d\mu) \to 0$.

# CHAPTER IV
# CONCLUSION

We have given natural generalizations for the Pettis integral and for the Bartle-Dunford-Schwartz integral by using finitely additive measures and in the latter case, by allowing the measures to take their values in a locally convex space.

We have given a natural generalization of the Bochner integral by allowing the functions to take their values in the set of bounded convex subsets of a locally convex space. Furthermore, we have shown the equivalence of this new integral and the Aumann integral under certain circumstances.

BIBLIOGRAPHY


[Ar72] Artstein, Z., Set Valued Measures, Trans. Amer. Math. Soc., 165(1972), 103-125.

[Au65] Aumann, R. J., Integrals of Set-Valued Functions, J. Math. Anal. App., 12(1965), 1-12.

[Ba55] Bartle, R. G., Dunford, N., and Schwartz, J., Weak Compactness and Vector Measures, Canad. J. Math., 7(1955), 289-305.

[Bi35] Birkhoff, G., Integration of Functions with Values in a Banach Space, Trans. Am. Math. Soc., 28(1935), 357-379.

[Bo60] Bourbaki, N., Topology Generale, Hermann Addison Wesley Paris Reading, Paris, 1960.

[Br68a] Brooks, J. K., An Integration Theory for Set-Valued Measures, I, Extrait du Bulletin Societe Royale des Sciences de Liege, No 5-8(1968), 312-319.

[Br68b] Brooks, J. K., An Integration Theory for Set-Valued Measures, II, Extrait du Bulletin Societe Royale des Sciences de Liege, No. 9-10(1968), 375-380.

[Br69] Brooks, J. K., Representations of Weak and Strong Integrals in Banach Spaces, Proc. Nat. Acad. Sci. USA, 63, No. 2(1969), 266-270.

[Br72] Brooks, J. K., Weak Compactness in the Space of Vector Measures, Bul. Amer. Math. Soc., 78(1972), 284-287.

[Br73] Brooks, J. K., Equicontinuous Sets of Measures and Applications to Vitali's Integral Convergence Theorem and Control Measures, Advances in Math., 10, No. 2(1973), 165-171.

[Bro70] Brooks, J. K., and Jewett, R. S., On Finitely Additive Vector Measures, Proc. Nat. Acad. Sci. USA, 67, No. 3(1970), 1294-1298.

[Ch47] Choquet, G., Convergences, Ann. Univ. Grenoble, 23(1947), 57-112.

[De65] Debrew, G., Integration of Correspondences, Fifth Berkeley Symposium on Math. Stat. and Prob., part 1(1965-1966), 351-372.

[De71] Debrew, G., and Schmeidler, D., The Radon-Nikodym Derivative of a Correspondence, Proc. Sixth Berkeley Sym. Math. Stat. and Prob., (1971), 41-56.

[Di77] Diestel, J., and Uhl, J., Vector Measures, American Math. Soc., Providence, Rhode Island, 1977.

[Dr76] Drewnowski, L., Additive and Countably Correspondences. Roczniki Polskiego Towarzystwa Matematiycznego, Seria I: Prace Matematyczne, 19(1976), 24-54.

[Du58] Dunford, N., and Schwartz, J., Linear Operators, Part I: General Theory, Interscience, New York, 1958.

[Dv50] Dvoretzky, A., Elimination of Randomization in Certain Problems of Statistics and the Theory of Games, Proc. Nat. Acad. Sci., 36(1950), 256-260.

[Ed65] Edwards, R., Functional Analysis, Holt, Rinehart and Winston, New York, 1965.

[Gr53] Grothendieck, A., Sur les Applications Linearies Faiblement Compactes d'Espaces du Type C(K), Canad. J. Math., 5(1953), 129-173.

[Ha74] Halmos, P. R., Measure Theory, Springer Verlag, New York, 1974.

[Hau62] Hausdorff, F., Megenlehre, Berlin-Leipsig, 1927, Translated by R. J. Aumann, Set Theory, Chelsea, 1962.

[He68] Hermes, J., Calculus of Set Val Functions and Control, J. Math. Mech., 18(1968-1969), 47-59.

[Hi27] Hill, L. S., Properties of Certain Aggregate Functions, Am. J. Math., 49(1927), 419-432.

[Ho54] Hormander, L., Sur la Fonction d'Appui des Ensembles Convexes dans un Espace Localement Convexe, Arkiv. for Math., 3(1954), 181-186.

[Ja69] Jacobs, M. Q., On the Approximation of Integrals of Multivalued Functions, SIAM J. Control, 7(1969), 158-177.

[Ka41] Kakutani, S., A Generalization of Browders Fixed Point Theorem, Duke Math. J., 8(1941), 457-459.

[Ko30] Kolmogorov, A., Untersuchungen Über den Integralbegriff, Math. Ann., 103(1930), 654-696.

[Ku54] Kudo, H., Dependent Experiments and Sufficient Statistics, Natur. Sci. Rep. Ochanomizu Univ., 4(1954), 151-163.

[Kur32] Kuratowski, K., Les Fonctions Semi-Continues dans l'Espace des Ensembles Fermes, Fund. Math., 18(1932), 148-160.

[Le70] Lewis, D., Integration with Respect to Vector Measures, Pacific J. Math., 33(1970), 157-165.

[Mi56a] Michael, E., Continuous Selections I, Ann. Math., 63(1956), 361-381.

[Mi56b] Michael, E., Continuous Selections II, Ann. Math., 63(1956), 562-580.

[Mo25] Moore, R. L., Concerning Upper Semi-Continuous Collections of Continua, Trans. Am. Math. Soc., 27(1925), 416-428.

[Moo22] Moore, E. H., and Smith, H. L., A General Theory of Limits, Am. J. Math., 44(1922), 102-121.

[Pe38] Pettis, B. J., On Integration in Vector Spaces, Trans. Am. Math. Soc., 44(1938), 277-304.

[Pr40] Price, G. B., Theory of Integration, Trans. Am. Math. Soc., 47(1940), 1-50.

[Ra52] Radstrom, H., An Embedding Theorem for Spaces of Convex Sets, Proc. Am. Math. Soc., 3(1952), 165-169.

[Ri63] Richter, M., Verallgemeinerung Eines in der Statistik Benotigten Satzes der Masstheorie, Math. Ann., 150(1963), 440-441.

[Ric42] Rickart, C. E., Integration in a Convex Linear Topological Space, Trans. Am. Math. Soc., 52(1942), 498-521.

[Ric44] Rickart, C. E., Decomposition of Additive Set Functions, Duke Math. Journal, 10(1944), 653-665.

[Ro69] Roberts, A. P., On Unconditional Convergence in Topological Vector Spaces, Proc. Royal Soc. Edinburg, Sect A.68(1969), 145-157.

[Sc71] Schmiedler, D., Convexity and Compactness in Countably Additive Correspondence in Differential Games and Related Topics, North-Holland, 1971.

[So85] Sousa, M. J., Set-Valued Integrals, Ph.D. Dissertation, University of Florida, Gainesville, 1985.

[vo35] von Neuman, J., On Complete Topological Spaces, Trans. Am. Math. Soc., 37(1935), 1-20.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James K. Brooks, Chairman
Professor of Mathematics

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